

# Stable Marriage with Covering Constraints: A Complete Computational Trichotomy

Matthias Mnich\*

Ildikó Schlotter†

## Abstract

We consider stable marriage problems equipped with *covering constraints*: here the input distinguishes a subset of women as well as a subset of men, and we seek a matching with fewest number of blocking pairs that matches all of the distinguished people. This concept strictly generalizes the notion of *arranged marriages* introduced by Knuth in 1976, in which the partner of each distinguished person is fixed a priori.

Our main result is a complete computational complexity trichotomy of the stable marriage problem with covering constraints, into polynomial-time solvable cases, fixed-parameter tractable cases, and cases that are  $W[1]$ -hard, for every choice among a set of natural parameters, namely the maximum length of preference lists for men and women, the number of distinguished men and women, and the number of blocking pairs allowed. Thereby, we fully answer an open problem of Hamada et al. (ESA 2011).

## 1 Introduction

The STABLE MARRIAGE (SM) problem is a fundamental matching problem first studied by Gale and Shapley [8] in 1962. We are given disjoint sets of men and women, and each person's preference list that orders the members of the other sex according to his/her preference. The aim is to find a stable matching, that is, a matching for which there exists no pair of a man and a woman who prefer each other to their partners given by the matching; such a pair is called a blocking pair. Gale and Shapley proved [8] that any instance of SM admits at least one stable matching, and gave a polynomial-time algorithm, known as the Gale-Shapley algorithm, to find one.

An early extension of the SM problem was suggested by Knuth [15] in 1976: *arranged marriages*. Here, a set  $Q^*$  of man-woman pairs is distinguished, and we seek a stable matching of the instance that contains  $Q^*$  as a subset. In contrast to the standard SM problem, a stable matching need not always exist in this setting with arranged marriages. Knuth [15] gave an algorithm that in time  $O(n^2)$  decides existence of a stable matching for instances of  $n$  people.

That work on arranged marriages has been extended in several ways. Gusfield and Irving [11] gave a rotation-based algorithm that decides existence of a stable matching in time  $O(|Q^*|^2)$  following  $O(n^4)$  preprocessing time. Dias et al. [5] additionally allow a set  $P^*$  of *forbidden marriages*, and show how to decide existence of a stable matching containing  $Q^*$  and disjoint from  $P^*$  in time  $O(m)$ , where  $m$  denotes the number of edges in the graph underlying the instance. If no such stable matching exists, one seeks a matching  $M$  extending  $Q^*$  and disjoint from  $P^*$  with fewest number of blocking pairs; Cseh and Manlove [3] showed that this problem is NP-hard to approximate within a factor  $O(n^{1-\varepsilon})$  for all  $\varepsilon > 0$ .

**SM with Covering Constraints.** We consider a more relaxed scenario: namely, a set  $W^*$  of women and a set  $M^*$  of men are distinguished, and a matching is feasible if it matches every person in  $W^* \cup M^*$ . In this scenario, feasible stable matchings need not exist, so we seek a feasible matching with the fewest number of blocking pairs. However, in contrast to Knuth's version, we do *not* specify to which partner the people

\*Universität Bonn, Bonn, German. [mmnich@uni-bonn.de](mailto:mmnich@uni-bonn.de). Supported by ERC Starting Grant 306465 (BeyondWorstCase).

†Budapest University of Technology and Economics, Budapest, Hungary. [ildi@cs.bme.hu](mailto:ildi@cs.bme.hu). Supported by the Hungarian National Research Fund (OTKA grants no. K-108383 and no. K-108947).

in  $\mathcal{W}^* \cup \mathcal{M}^*$  are matched to. We call this problem STABLE MARRIAGE WITH COVERING CONSTRAINTS (SMC). Clearly, SMC can be reduced to SM with arranged marriages, via a Turing reduction, by guessing for each person in  $\mathcal{W}^* \cup \mathcal{M}^*$  their “correct” partner in time  $O((\Delta^*)^{|\mathcal{W}^* \cup \mathcal{M}^*|})$ , where  $\Delta^*$  is an upper bound on the length of the preference lists of people in  $\mathcal{W}^* \cup \mathcal{M}^*$ .

We remark that it is easy to decide for a given instance if it admits a feasible *stable* matching (i.e., without any blocking pairs): the reason is that the set of women whose covering constraint can be satisfied is identical for any stable matching, by the Rural Hospitals Theorem [9]. For the same reason, it is easy to find a stable matching that violates the fewest number of covering constraints.

Hamada et al. [12] studied the HOSPITALS/RESIDENTS WITH LOWER QUOTAS (HRLQ) problem (defined more precisely in Sect. 2), which is a many-to-one generalization of SMC with covering constraints defined only on one side (say, for women). In HRLQ, covering constraints take the form of quota lower bounds, imposed by hospitals on the minimum number of residents they need to accommodate. The results by Hamada et al. [12] imply that SMC cannot be approximated within a factor of  $(|\mathcal{M}| + |\mathcal{W}|)^{1-\varepsilon}$  for any  $\varepsilon > 0$  unless  $P = NP$ , even if all women have the same preference list. Further, they gave an exponential-time exact algorithm that in time  $O((|\mathcal{W}||\mathcal{M}|)^{b+1})$  decides for a given instance of SMC with one-sided covering constraints whether it admits a feasible matching with at most  $b$  blocking pairs. Since the degree of the polynomial in the run time of their algorithm grows with  $b$ , theirs is not a fixed-parameter algorithm parameterized by  $b$ . This led them to propose as an open question whether HRLQ (and, in particular, SMC) is fixed-parameter tractable parameterized by the minimum number  $b$  of blocking pairs.

## 1.1 Our Results

In this paper we answer Hamada et al.’s question [12] in a strong sense. We not only resolve the parameterized complexity of SMC with respect to the minimum number of blocking pairs, but we provide a complete computational complexity trichotomy with respect to an entire set of natural parameters of SMC.

The parameters we consider are

- (i) the minimum number  $b$  of blocking pairs in any feasible matching of a given instance,
- (ii) the number  $|\mathcal{W}^*|$  of women with covering constraint,
- (iii) the number  $|\mathcal{M}^*|$  of men with covering constraint,
- (iv) the maximum length  $\Delta_{\mathcal{W}}$  of women’s preference lists, and
- (v) the maximum length  $\Delta_{\mathcal{M}}$  of men’s preference lists.

Each parameter  $p \in \{b, |\mathcal{W}^*|, |\mathcal{M}^*|, \Delta_{\mathcal{M}}, \Delta_{\mathcal{W}}\}$  can either appear in the exponent of the run time (such as  $n^{O(p)}$ ), in the multiplier of the run time (such as  $f(p) \cdot n^{O(1)}$ ), or not appear at all in the run time. Therefore, there are  $3^5 = 243$  corresponding complexity questions for SMC. The following theorem answers all these questions:

**Theorem 1.** *For any subset  $S \subseteq \{b, |\mathcal{W}^*|, |\mathcal{M}^*|, \Delta_{\mathcal{M}}, \Delta_{\mathcal{W}}\}$  and any choice of elements in  $S$  as “constant”, “parameter” or “unbounded”, SMC is either in  $P$ , or NP-hard and fixed-parameter tractable, or NP-hard and  $W[1]$ -hard, as shown in Table 1.*

We show that every question arising in Theorem 1 can be answered by a set of 8 positive results and a set of 4 negative results; these are given in Table 1. We provide a decision diagram in Appendix A, to show they indeed cover all possible cases. In particular, our trichotomy result allows us to answer to Hamada et al.’s question [12] is negative, in that there is no fixed-parameter algorithm for SMC with parameter  $b$  unless  $FPT = W[1]$ :

**Theorem 2.** *SMC-1 is  $W[1]$ -hard parameterized by  $b + |\mathcal{W}^*|$ , even if there is a master list over men as well as one over women,  $\Delta_{\mathcal{M}} = 3$ ,  $\Delta_{\mathcal{W}} = 3$  and each woman with covering constraint finds only a single man acceptable.*

constants	parameters	complexity
$ \mathcal{M}^*  = 0,  \mathcal{W}^*  = 0$		poly (Gale-Shapley alg.)
$ \mathcal{M}^*  = 0,  \mathcal{W}^* , \Delta_{\mathcal{M}}$		poly (Theorem 9)
$ \mathcal{M}^* ,  \mathcal{W}^* , \Delta_{\mathcal{M}}, \Delta_{\mathcal{W}}$		poly (Theorem 10)
$ \mathcal{M}^*  = 0, \Delta_{\mathcal{M}} \leq 2$		poly (Theorem 11)
$\Delta_{\mathcal{W}} \leq 2, \Delta_{\mathcal{M}} \leq 2$		poly (Observ. 2)
$b$		poly (Observ. 1)
$ \mathcal{M}^*  = 0, \Delta_{\mathcal{W}} \leq 2, \Delta_{\mathcal{M}} \geq 3$		NP-hard (Theorem 14)
$ \mathcal{W}^*  = 1, \Delta_{\mathcal{W}} \leq 2, \Delta_{\mathcal{M}} \geq 3$		NP-hard (Theorem 15)
$ \mathcal{M}^*  = 0, \Delta_{\mathcal{W}} \geq 3, \Delta_{\mathcal{M}} \geq 3, \Delta^* = 1$	$b +  \mathcal{W}^* $	W[1]-hard (Theorem 3)
$ \mathcal{M}^*  = 0,  \mathcal{W}^*  \geq 1, \Delta_{\mathcal{W}} \geq 3, \Delta^* = 1$	$b + \Delta_{\mathcal{M}}$	W[1]-hard (Theorem 7)
$\Delta_{\mathcal{W}} \leq 2$	$ \mathcal{W}^*  +  \mathcal{M}^* $	FPT (Theorem 16)
$\Delta_{\mathcal{W}} \leq 2$	$b$	FPT (Cor. 20)

Table 1: Summary of our results for STABLE MARRIAGE WITH COVERING CONSTRAINTS.

## 1.2 Related Work

The HRLQ problem recently gained quite some interest from the algorithmic community [1, 7, 12, 14, 16]. There are different models of penalty function when these quota lower bounds cannot be satisfied by any matching of the instance. Hamada et al. [12] simply impose some technical conditions that this case never occurs and a feasible matching always exists, and they then seek a feasible matching with fewest number of blocking pairs. In contrast, the model of Biró et al. [1] allows some hospitals to be closed, i.e., to receive no residents; they prove that the problem of asking for a feasible solution is NP-complete.

Huang [14] considers stable matchings in which not only individual hospitals but also subsets of hospitals declare quota upper and lower bounds; his model finds applications in academic hiring. He proved that deciding the existence of a stable matching is in P if the family of the sets of hospitals that declare quota bounds have a certain structural property, and is NP-complete otherwise. This work was followed up by Fleiner and Kamiyama [7], who used matroid tools to solve a generalization of Huang’s model and gave a polyhedral description of the stable assignments. Recently, Yokoi [16] pushed these results to an even more general model of stable matchings with quota lower bounds.

**Organization.** After the preliminaries, we start with the main intractability result in Sect. 3, which answers Hamada et al.’s question in a strong sense. This result shows that we have three possible directions to achieve tractability: (i) to lower  $b$  to be a constant, (ii) to lower  $|\mathcal{W}^*|$  to be a constant, or (iii) to lower either  $\Delta_{\mathcal{W}}$  or  $\Delta_{\mathcal{M}}$  to 2. We cover the cases (i) and (ii) in Sect. 5, and case (iii) in Sect. 6. In addition, Sect. 4 provides polynomial-time approximation results for HRLQ and SMC, used also in the polynomial-time algorithms of Sect. 5.

## 2 Preliminaries

An instance  $I$  of the STABLE MARRIAGE (SM) problem consists of a set  $\mathcal{M}$  of men and a set  $\mathcal{W}$  of women. Each person  $x \in \mathcal{M} \cup \mathcal{W}$  has a preference list  $L(x)$  that strictly orders the members of the other party acceptable for  $x$ . We thus write  $L(x)$  as a vector  $L(x) = (y_1, y_2, \dots, y_t)$ , denoting that  $y_i$  is (strictly) *preferred* by  $x$  over  $y_j$  for each  $i$  and  $j$  with  $1 \leq i < j \leq t$ . A *matching*  $M$  for  $I$  is a set of man-woman pairs appearing in each other’s preference lists such that each person is contained in at most one pair of  $M$ ; some persons may be left unmatched by  $M$ . For each person  $x$  we denote by  $M(x)$  the person assigned by  $M$  to  $x$ . For a matching  $M$ , a man  $m$  and a woman  $w$  included in each other’s preference lists form a *blocking pair* if

(i)  $m$  is either unmatched or prefers  $w$  to  $M(m)$ , and (ii)  $w$  is either unmatched or prefers  $m$  to  $M(w)$ . In the STABLE MARRIAGE WITH COVERING CONSTRAINTS (SMC) problem, we are given additional subsets  $\mathcal{W}^* \subseteq \mathcal{W}$  and  $\mathcal{M}^* \subseteq \mathcal{M}$  of *distinguished* people that must be matched; a matching  $M$  is *feasible* if it matches everybody in  $\mathcal{W}^* \cup \mathcal{M}^*$ . The objective of SMC is to find a feasible matching for  $I$  with minimum number of blocking pairs. If only people from one gender are distinguished, then without loss of generality, we assume these to be women; this special case will be denoted by SMC-1.

The many-to-one extension of SMC-1 is the HOSPITALS/RESIDENTS WITH QUOTA LOWER BOUNDS (HRLQ) problem whose input consists of a set  $\mathcal{R}$  of residents and a set  $\mathcal{H}$  of hospitals that have ordinal preferences over the acceptable members of the other party. Each hospital  $h \in \mathcal{H}$  has a quota lower bound  $\underline{q}(h)$  and quota upper bound  $\overline{q}(h)$ , which bound the number of residents that can be assigned to  $h$  from below and above. One seeks an *assignment*  $M$  that maps a subset of the residents to hospitals that respects acceptability and is *feasible*, that is,  $\underline{q}(h) \leq |M(h)| \leq \overline{q}(h)$  for each hospital  $h$ . Here,  $M(h)$  is a set of residents assigned to some  $h \in \mathcal{H}$ . We say that a hospital  $h$  is *under-subscribed* if  $|M(h)| < \overline{q}(h)$ . For an assignment  $M$  of an instance of HRLQ, a pair  $\{r, h\}$  of a resident  $r$  and a hospital  $h$  is *blocking* if (i)  $r$  is unassigned or prefers  $h$  to the hospital assigned to  $r$  by  $M$ , and (ii)  $h$  is under-subscribed or prefers  $r$  to one of the residents in  $M(h)$ . The task in HRLQ is to find a feasible assignment with minimum number of blocking pairs.

Some instances of SMC may admit a *master list* over women, which is a total ordering  $L_{\mathcal{W}}$  of all women, such that for each man  $m \in \mathcal{M}$ , the preference list  $L(m)$  is the restriction of  $L_{\mathcal{W}}$  to those women that  $m$  finds acceptable. Similarly, we consider master lists over men.

With each instance  $I$  of SMC (or HRLQ) we can naturally associate a bipartite graph  $G_I$  whose vertex partitions correspond to  $\mathcal{M}$  and  $\mathcal{W}$  (or  $\mathcal{R}$  and  $\mathcal{H}$ ), respectively, and there is an edge between a man  $m \in \mathcal{M}$  and a woman  $w \in \mathcal{W}$  (or between a resident  $r \in \mathcal{R}$  and a hospital  $h \in \mathcal{H}$ , respectively) if they appear in each other's preference lists. We may refer to entities of  $I$  as vertices, or a pair of entities as edges, without mentioning  $G_I$  explicitly.

**Parameterized complexity.** The framework of parameterized complexity deals with computationally hard problems, examining their complexity in a more detailed way than classical complexity theory. In a parameterized problem  $\Pi$ , each input instance  $I$  is associated with an integer  $k$  called the *parameter*. An algorithm which decides instances  $I$  of  $\Pi$  in time  $f(k) \cdot |I|^{O(1)}$  for some computable function  $f$  is called a *fixed-parameter algorithm*. Note that the dependence of the polynomial in the run time is constant, but the dependence on the parameter  $k$  can be arbitrary (and is typically exponential). However, if the parameter of a given instance is small, then such an algorithm can be useful in practice even if the overall size of the instance is large.

The class of problems admitting fixed-parameter algorithms is denoted by FPT. To argue that a problem is *not* in FPT, parameterized complexity provides a hardness theory. For two parameterized problems  $\Pi_1$  and  $\Pi_2$ , a *parameterized reduction* from  $\Pi_1$  to  $\Pi_2$  is a function  $f$ , computable by a fixed-parameter algorithm, that maps each instance  $(I_1, k_1)$  of  $\Pi_1$  to an instance  $f(I_1, k_1) = (I_2, k_2)$  of  $\Pi_2$  such that (i)  $(I_1, k_1)$  is a yes-instance of  $\Pi_1$  if and only if  $(I_2, k_2)$  is a yes-instance of  $\Pi_2$ , and (ii)  $k_2 \leq g(k_1)$  for some function  $g$ . The basic class of parameterized intractability is W[1]: proving a problem  $\Pi$  to be W[1]-hard is strong evidence that  $\Pi \notin \text{FPT}$ . In this paper, we will always prove W[1]-hardness by providing a parameterized reduction from the MULTICOLORED CLIQUE problem: given a graph  $G$  with its vertex set partitioned into sets  $V_1, \dots, V_k$ , decide whether there exists a clique of size  $k$  in  $G$  that has one vertex from each of the sets  $V_i$ ; the parameter is the integer  $k$ . Since MULTICOLORED CLIQUE is W[1]-hard [6], a parameterized reduction from it to some parameterized problem  $\Pi$  implies W[1]-hardness of  $\Pi$  as well.

For more on parameterized complexity, we refer the reader to the recent monograph by Cygan et al. [4].

### 3 Strong Parameterized Intractability of SMC

This section provides parameterized intractability and inapproximability results for SMC showing the hardness of finding feasible matchings with minimum number of blocking pairs. Namely, we prove SMC-1 to be

$W[1]$ -hard parameterized by the number  $b$  of blocking pairs we aim for plus the number  $|\mathcal{W}^*|$  of women that must be covered, even in a very restricted setting.

**Theorem 3.** *SMC-1 is  $W[1]$ -hard parameterized by  $b + |\mathcal{W}^*|$ , even if there is a master list over men as well as one over women, all preference lists are of length at most 3, and  $|L(w)| = 1$  for each woman  $w \in \mathcal{W}^*$ .*

*Proof.* We give a reduction from the  $W[1]$ -hard MULTICOLORED CLIQUE parameterized by the size of the solution [6]. Let  $G$  be the input graph, with its vertex set partitioned into  $k$  sets  $V_1, \dots, V_k$ ; the task is to find a clique of size  $k$  in  $G$  containing exactly one vertex from each of the sets  $V_i$ . We let  $E_{i,j}$  denote those edges that run between  $V_i$  and  $V_j$  for some  $1 \leq i < j \leq k$ . We fix an ordering on the vertices and edges of  $G$  that places vertices of  $V_i$  before vertices of  $V_j$  whenever  $i < j$ . We will write  $\text{succ}(x)$  to denote the vertex following  $x$  in this ordering, and we let  $v_i^1$  and  $v_i^\infty$  denote the first and last vertices in  $V_i$ , respectively. Similarly, we write  $\text{succ}(\{x, y\})$  for the edge following  $\{x, y\}$ , and we let  $e_{i,j}^1$  and  $e_{i,j}^\infty$  denote the first and last edges in  $E_{i,j}$ , respectively. We will also write  $\text{pred}(x)$  and  $\text{pred}(\{x, y\})$  for the predecessor of  $x$  or  $\{x, y\}$ , respectively. Also, we denote the  $h$ -th neighbor of some vertex  $x$  as  $n(x, h)$ . For simplicity, we assume that there is no isolated vertex in  $G$ .

We construct an instance  $I$  of SMC as follows; see Fig. 1 and Fig. 2 for an illustration.

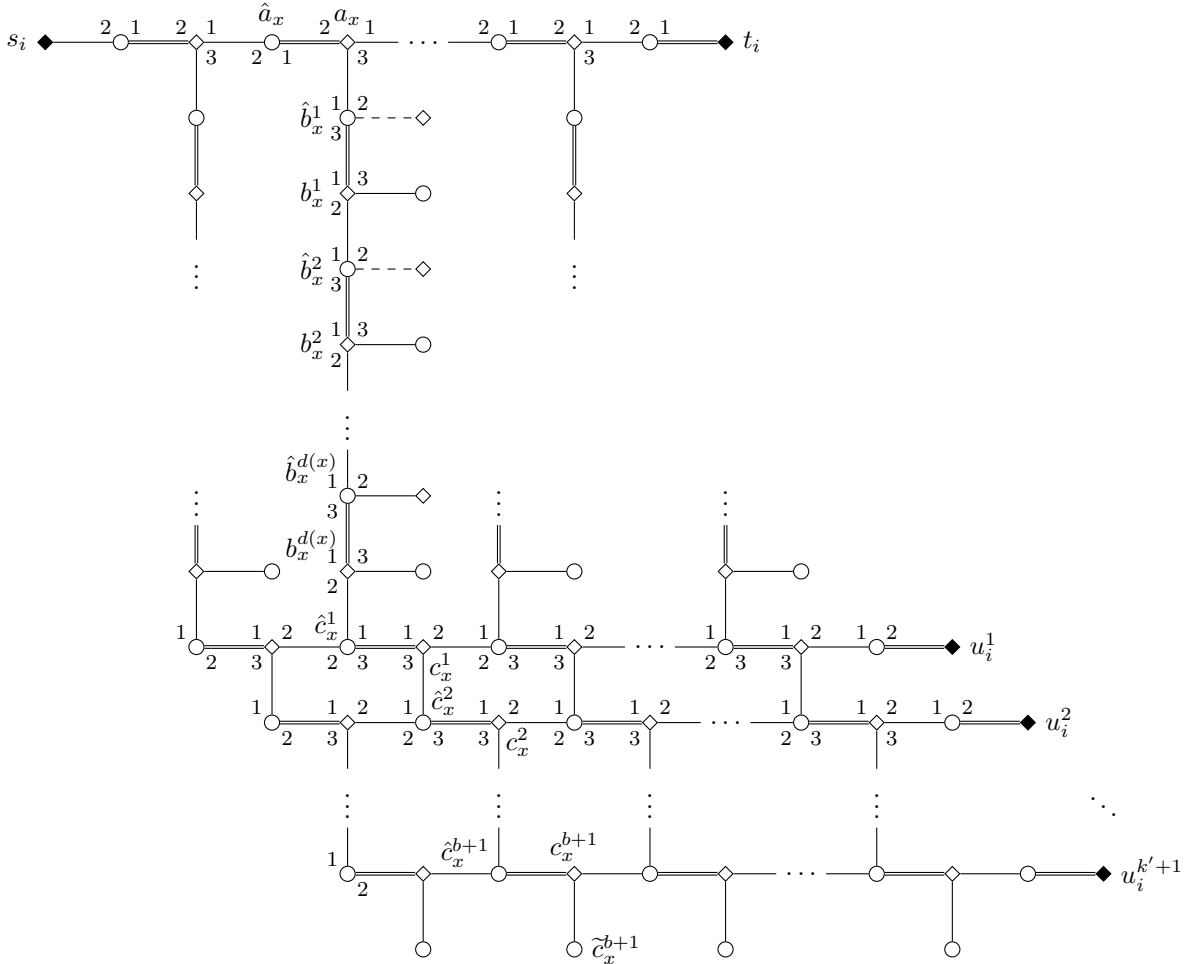


Figure 1: Node selecting gadget  $G_i$  in the proof of Theorem 3.



Table 2: Preference lists of women and men in node selecting gadgets.

$L(a_x)$	$= (\hat{a}_{\text{succ}(x)}, \hat{a}_x, \hat{b}_x^1),$	where $x \in V_i \setminus \{v_i^\infty\},$
$L(a_x)$	$= (\hat{t}_i, \hat{a}_x, \hat{b}_x^1),$	where $x = v_i^\infty,$
$L(b_x^h)$	$= (\hat{b}_x^h, \hat{b}_x^{h+1}, \tilde{b}_x^h),$	
$L(c_x^h)$	$= (\hat{c}_x^h, \hat{c}_{\text{succ}(x)}^h, \hat{c}_x^{h+1}),$	where $1 \leq h \leq b, x \in V_i \setminus \{v_i^\infty\},$
$L(c_x^h)$	$= (\hat{c}_x^h, \hat{u}_i^h, \hat{c}_x^{h+1}),$	where $1 \leq h \leq b$ and $x = v_i^\infty,$
$L(c_x^{b+1})$	$= (\hat{c}_x^{b+1}, \hat{c}_{\text{succ}(x)}^h, \tilde{c}_x^{b+1}),$	where $x \in V_i \setminus \{v_i^\infty\},$
$L(c_x^{b+1})$	$= (\hat{c}_x^{b+1}, \hat{u}_i^{b+1}, \tilde{c}_x^{b+1}),$	where $x = v_i^\infty,$
$L(s_i)$	$= (\hat{a}_x),$	where $x = v_i^1,$
$L(t_i)$	$= (\hat{t}_i),$	
$L(u_i^h)$	$= (\hat{u}_i^h),$	
$L(\hat{a}_x)$	$= (a_x, a_{\text{pred}(x)}),$	where $x \in V_i \setminus \{v_i^1\},$
$L(\hat{a}_x)$	$= (a_x, s_i),$	where $x = v_i^1,$
$L(\hat{b}_x^h)$	$= (b_x^{h-1}, b_{x \rightarrow y}, b_x^h),$	where $y = n(x, h), x \in V_i, y \in V_j$ and $i < j$
$L(\hat{b}_x^h)$	$= (b_x^{h-1}, b_{y \rightarrow x}, b_x^h),$	where $y = n(x, h), x \in V_i, y \in V_j$ and $i > j$
$L(\hat{c}_x^h)$	$= (c_x^{h-1}, c_{\text{pred}(x)}^h, c_x^h),$	where $x \in V_i \setminus \{v_i^1\},$
$L(\hat{c}_x^h)$	$= (c_x^{h-1}, c_x^h),$	where $x = v_i^1,$
$L(\hat{t}_i)$	$= (t_i, a_x),$	where $x = v_i^\infty,$
$L(\hat{u}_i^h)$	$= (c_x^h, u_i^h),$	where $x = v_i^\infty,$
$L(\tilde{w})$	$= (w),$	for any dummy woman $\tilde{w}.$

Table 3: Preference lists of women and men in edge selecting gadgets.

$L(a_{\{x,y\}})$	$= (\hat{a}_{\text{succ}(\{x,y\})}, \hat{a}_{\{x,y\}}, \hat{b}_{x \rightarrow y}),$	where $\{x, y\} \in E_{i,j} \setminus \{e_{i,j}^\infty\}$ and $x$ precedes $y,$
$L(a_{\{x,y\}})$	$= (\hat{t}_{i,j}, \hat{a}_{\{x,y\}}, \hat{b}_{x \rightarrow y}),$	where $\{x, y\} = e_{i,j}^\infty$ and $x$ precedes $y,$
$L(b_{x \rightarrow y})$	$= (\hat{b}_{x \rightarrow y}, \hat{b}_x^h, \hat{b}_{y \rightarrow x}),$	where $y = n(x, h)$ and $x$ precedes $y$ in $V(G),$
$L(b_{y \rightarrow x})$	$= (\hat{b}_{y \rightarrow x}, \hat{b}_y^h, \tilde{b}_{y \rightarrow x}),$	where $y = n(x, h)$ and $x$ precedes $y$ in $V(G),$
$L(s_{i,j})$	$= (\hat{a}_{\{x,y\}}),$	where $\{x, y\} = e_{i,j}^1,$
$L(t_{i,j})$	$= (\hat{t}_{i,j}),$	
$L(\hat{a}_{\{x,y\}})$	$= (a_{\{x,y\}}, a_{\text{pred}(\{x,y\})}),$	where $\{x, y\} \in E_{i,j} \setminus \{e_{i,j}^1\},$
$L(\hat{a}_{\{x,y\}})$	$= (a_{\{x,y\}}, s_{i,j}),$	where $\{x, y\} = e_{i,j}^1,$
$L(\hat{b}_{x \rightarrow y})$	$= (a_{\{x,y\}}, b_{x \rightarrow y}),$	where $x$ precedes $y$ in $V(G),$
$L(\hat{b}_{y \rightarrow x})$	$= (b_{x \rightarrow y}, b_{y \rightarrow x}),$	where $x$ precedes $y$ in $V(G),$
$L(\hat{t}_{i,j})$	$= (t_{i,j}, a_{\{x,y\}}),$	where $\{x, y\} = e_{i,j}^\infty,$
$L(\tilde{w})$	$= (w),$	for any dummy woman $\tilde{w}.$

The master list over men is derived from  $L_W$  by letting  $\hat{w}_1$  precede  $\hat{w}_2$  whenever  $w_1$  precedes  $w_2$  in  $L_W$ , and adding all dummies at the end in an arbitrary order. It is easy to check that the preference lists given in Tables 2 and 3 are indeed compatible with these master lists. This completes the construction of the instance.

We are going to prove that the constructed instance  $I$  admits a feasible assignment with at most  $b$  blocking pairs if and only if there is a clique of size  $k$  in the graph  $G$ .

" $\Rightarrow$ ": Suppose there is a feasible matching  $M$  of men to women with at most  $b$  blocking pairs. Let  $G_\Delta$  be the symmetric difference  $M \Delta M_s$ . Notice that for each woman  $s \in S$ , the difference  $G_\Delta$  must contain exactly one path containing  $s$  as its endpoint, since the women in  $S$  must be matched in  $M$ , but are unmatched in  $M_s$ . Similarly, no path of  $G_\Delta$  can contain a woman in  $T \cup U$ . We call a path  $P$  in  $G_\Delta$  with an endpoint  $s$  in  $S$  an *augmenting path*. We say that  $P$  starts at  $s$  and ends at its other endpoint, and we refer to that path starting at  $s_i$  (or  $s_{i,j}$ ) as  $P_i$  (or  $P_{i,j}$ , respectively).

We define the *cost* of some path  $P$  of  $G_\Delta$  as the number of blocking pairs  $\{m, w\}$  for  $M$  involving a woman  $w$  that appears on  $P$ . Since  $M_s$  is stable, it should be clear that each augmenting path contains at least one edge that is blocking for  $M$ , so each path in  $G_\Delta$  has cost at least 1. As there are exactly  $k + \binom{k}{2}$  augmenting paths (as all women in  $S$  must be matched by  $M$ ), we get a minimum cost of  $k + \binom{k}{2}$ . Note also that the total cost of all paths in  $G_\Delta$  cannot exceed  $b = 2k + \binom{k}{2}$ . Claim 1 is therefore crucial.

**Claim 1.** *The following holds for any augmenting path  $P$  of  $G_\Delta$ :*

- $P$  cannot end at a dummy  $\tilde{c}_x^{b+1}$  for some  $x \in V(G)$ .
- $P$  contains an edge  $\{a, \hat{a}\}$  for some  $a \in A \cup A'$  that blocks  $M$ .
- If  $P$  is not disjoint from  $G_i$  for some  $i$ , then  $P$  has cost at least 2.

*Proof of Claim 1.* To prove (a), suppose for contradiction that  $P$  ends at  $\tilde{c}_x^{b+1}$ , where  $x \in V_i$ . Clearly,  $P$  must contain at least one woman from each of the  $b+1$  sets  $\{c_v^h \mid v \in V_i\}$ ,  $h = 1, \dots, b+1$ . Fix  $h$ , and let us consider the last  $v \in V_i$  for which  $c_v^h$  is incident to an edge of  $G_\Delta$ . Let  $w = c_{\text{succ}(v)}^h$  if  $v \neq v_i^\infty$ , or otherwise let  $w = u_i^h$ . Then the edge  $\{c_v^h, w\}$  yields a blocking pair in  $M$ , as  $M(w) = M_s(w) = \hat{w}$ , and thus  $\hat{w}$  prefers  $c_v^h$  to  $w$ . This reasoning gives us  $b+1$  different blocking pairs for  $M$ , one for each index  $h$ , contradicting our assumption on  $M$ .

To prove (b), let us consider the case when  $P = P_i$  for some  $i$ ; the argument goes the same way for the case where  $P = P_{i,j}$  for some  $i$  and  $j$ . If  $P$  ends at  $a_x$  for some  $x \in V_i$ , then  $a_x$  forms a blocking pair with  $\hat{a}_x$  in  $M$ . If  $P$  does not end at a woman in  $A$ , then it must contain the edge  $\{a_x, b_x^1\}$  for some  $x$ , in which case  $\{a_x, \hat{a}_x\}$  is again blocking in  $M$ , showing (b).

To see (c), first observe that if  $P$  is not disjoint from  $G_i$ , then  $P$  ends in  $G_i$ , simply because of its property that it contains edges from  $M$  and  $M_s$  in an alternating fashion. Therefore, the last woman  $w$  on  $P$  must be in  $B \cup C$ . If  $w = b_x^h$  for some  $b \in B$ , then the edge  $\{b_x^h, \hat{b}_x^{h+1}\}$  is blocking  $M$ , as  $b_x^h$  cannot get its first choice  $\hat{b}_x^h$  in  $M$  (and  $\hat{b}_x^{h+1}$  cannot be on  $P$ , as that would imply that  $b_x^{h+1}$  is on  $P$ , contradicting the choice of  $w$ ). If, by contrast,  $w = c$  for some  $c \in C$ , then  $P$  must end at  $w$  by (a), and then  $c$  forms a blocking pair with the third man in its preference list (for whom  $c$  is the first choice). In either case,  $w$  is involved in a blocking pair, which together with the blocking pair guaranteed by (b) implies that  $P$  has cost at least 2.

This completes the proof of Claim 1.  $\square$

Claim 1 proves that for each  $i \in \{1, \dots, k\}$  the augmenting path  $P_i$  has cost at least 2. Since all the remaining  $\binom{k}{2}$  augmenting paths have cost at least 1, and the total cost of these paths must be at most  $b = 2k + \binom{k}{2}$ , we get that any path  $P_i$  (or  $P_{i,j}$ ) must have cost *exactly* 2 (or 1, respectively). Furthermore, it also follows that no other path of  $G_\Delta$  can enter or start in  $G_i$ , for any  $i$ , as that would imply that the number of blocking pairs for  $M$  is more than  $b$ . In addition, it is not hard to see that  $G_\Delta$  does not contain any cycle, because all cycles in the graph underlying  $I$  contain two consecutive edges not in  $M_s$ . Hence, it follows that the only connected component in  $G_\Delta$  that is not disjoint from  $G_i$  is  $P_i$ .



To deal with the possibly courses the path  $P_i$  may take for some  $1 \leq i \leq k$ , let  $x_i$  denote the vertex in  $V_i$  for which  $\{a_{x_i}, \hat{a}_{x_i}\}$  is the blocking edge guaranteed by statement (b) of Claim 1. Observe that  $P_i$  either ends at  $a_{x_i}$  or contains the edge  $\{a_{x_i}, \hat{b}_{x_i}^1\}$ . In either case, we say that  $P_i$  *selects*  $x_i$  from  $V_i$ ; clearly, there can be only one vertex in  $V_i$  selected by  $P_i$ .

Consider now  $P_{i,j}$  for some  $1 \leq i < j \leq k$ . Recall that  $P_{i,j}$  has cost 1. Therefore, statement (b) of Claim 1 proves that the only blocking edge incident to some woman on  $P_{i,j}$  must be  $\{a_{\{x,y\}}, \hat{a}_{\{x,y\}}\}$  for some  $\{x,y\} \in E_{i,j}$ . We say that  $P_{i,j}$  *selects* the edge  $\{x,y\}$ ; without loss of generality, let us assume that  $x$  precedes  $y$ . By statement (c) of Claim 1, we also know that  $P_{i,j}$  cannot leave  $G_{i,j}$ , which means that it can only have cost 1 if it ends at  $\tilde{b}_{y \rightarrow x}$ . In particular, it contains the edges  $\{b_{x \rightarrow y}, \hat{b}_{y \rightarrow x}\}$  and  $\{b_{y \rightarrow x}, \tilde{b}_{y \rightarrow x}\}$ . Observe that the edge  $\{b_{x \rightarrow y}, \hat{b}_{x \rightarrow y}^h\}$  where  $h$  is such that  $y = n(x, h)$  cannot be blocking in  $M$  (as this would indicate a cost of 2 for  $P_{i,j}$ ), yielding that  $\hat{b}_x^h$  must be matched to  $b_x^{h-1}$  in  $M$ . By the arguments of the previous paragraph, this means that  $P_i$  must contain the subpath  $(a_x, \hat{b}_x^1, b_x^1, \dots, \hat{b}_x^h, b_x^h)$ . Hence, we obtain that  $x$  must be selected by  $P_i$ . Similarly, from the fact that the edge  $\{b_{y \rightarrow x}, \hat{b}_y^\ell\}$  where  $x = n(y, \ell)$  is not blocking in  $M$  we get that  $y$  must be selected by  $P_j$ .

Thus, we obtain that if an edge is selected by  $P_{i,j}$  for some  $i$  and  $j$ , then its endpoints must be selected by  $P_i$  and  $P_j$ . As this must hold for each pair of indices with  $1 \leq i < j \leq k$ , we obtain that there must be  $\binom{k}{2}$  edges in  $G$  whose endpoints are among the  $k$  selected vertices. This can only happen if these edges are the edges of a clique of size  $k$ .

" $\Leftarrow$ ": Suppose now that  $G$  has a clique of size  $k$  formed by the vertices  $x_1, \dots, x_k$ , with  $x_i \in V_i$  for each  $i \in \{1, \dots, k\}$ . Instead of directly defining the required matching  $M$  that is feasible and admits at most  $b$  blocking pairs, we give  $M_s \triangle M$  as the union of paths  $P_i$ ,  $i \in \{1, \dots, k\}$ , and paths  $P_{i,j}$ ,  $1 \leq i < j \leq k$ , defined as follows.

We set  $P_i$  as the path

$$P_i = (s_i, \hat{a}_{v_i^1}, a_{v_i^1}, \dots, \hat{a}_{x_i}, a_{x_i}, \hat{b}_{x_i}^1, b_{x_i}^1, \dots, \hat{b}_{x_i}^{d(x_i)}, b_{x_i}^{d(x_i)}, \tilde{b}_{x_i}^{d(x_i)}) .$$

Similarly, we define

$$P_{i,j} = (s_{i,j}, \hat{a}_{e_{i,j}^1}, a_{e_{i,j}^1}, \dots, \hat{a}_{\{x_i, x_j\}}, a_{\{x_i, x_j\}}, \hat{b}_{x_i \rightarrow x_j}, b_{x_i \rightarrow x_j}, \hat{b}_{x_j \rightarrow x_i}, b_{x_j \rightarrow x_i}, \tilde{b}_{x_j \rightarrow x_i}) .$$

It is straightforward to verify that the blocking pairs for  $M$  are then the  $k$  edges  $\{a_{x_i}, \hat{a}_{x_i}\}$ ,  $i \in \{1, \dots, k\}$ , the  $k$  edges  $\{b_{x_i}^{d(x_i)}, c_{x_i}^1\}$ , and the  $\binom{k}{2}$  edges  $\{a_{\{x_i, x_j\}}, \hat{a}_{\{x_i, x_j\}}\}$ ,  $1 \leq i < j \leq k$ . The feasibility of  $M$  is trivial; this completes the proof of Theorem 3.  $\square$

Assuming ETH, CLIQUE was shown not to admit algorithms giving the correct answer in time  $f(k) \cdot n^{o(k)}$  for all  $n$ -vertex instances and any computable function  $f$  [2, Theorem 5.4]. The known reduction from CLIQUE to MULTICOLORED CLIQUE does not change the parameter [6]. Finally, in the proof of Theorem 3, an instance of MULTICOLORED CLIQUE with solution size  $k$  is reduced to an instance of SMC-1 with parameter  $b = O(k^2)$ .

**Corollary 4.** *If ETH holds, SMC-1 cannot be solved in time  $f(b) \cdot n^{o(\sqrt{b})}$  for any computable function  $f$ , even if there is a master list over men and over women, all preference lists have length at most 3, and each woman in  $\mathcal{W}^*$  finds only a single man acceptable.*

## 4 Polynomial-Time Approximation for HRLQ

Here we provide a polynomial-time algorithm that yields a constant-factor approximation for instances of HRLQ where both the maximum length  $\Delta_{\mathcal{R}}$  of residents' preference lists and the total sum  $\mathbf{q}_{\Sigma}$  of all lower quotas is constant. Our objective is to find an assignment that satisfies all quota lower and upper bounds and minimizes the number of blocking pairs.

**Theorem 5.** *Let  $I$  be an instance  $I$  of HRLQ, and  $\mathbf{q}_{\Sigma}$  the sum of lower quota bounds taken over all hospitals in  $I$ . There is an algorithm that in polynomial time either outputs a feasible assignment for  $I$  with at most  $(\Delta_{\mathcal{R}} - 1)\mathbf{q}_{\Sigma}$  blocking pairs, involving only  $\mathbf{q}_{\Sigma}$  residents, or concludes that no feasible assignment exists.*

*Proof.* We start by finding an assignment  $M_q$  that assigns  $\underline{q}(h)$  residents to each hospital  $h \in \mathcal{H}^*$ , and has the following property:

*for each hospital  $h \in \mathcal{H}^*$ , all residents that are preferred by  $h$  to the least preferred resident in  $M_q(h)$  are contained in  $\bigcup_{h' \in \mathcal{H}^* \setminus \{h\}} M_q(h')$ .* (†)

Such an assignment can be obtained as follows. We start from an arbitrary assignment  $M$  that assigns  $\underline{q}(h)$  residents to each  $h \in \mathcal{H}^*$  (if no such assignment exists, then we can stop and reject); such an assignment, if existent, can be found in polynomial time by an algorithm of Hopcroft and Karp [13]. Then we greedily re-assign residents to hospitals of  $\mathcal{H}^*$ , one-by-one: at each step, we take a hospital  $h \in \mathcal{H}^*$ , and if there exists a resident  $r$  not assigned to any other hospital in  $\mathcal{H}^*$  that  $h$  prefers to the least preferred resident  $r'$  in  $M(h)$ , then we replace  $r'$  with  $r$  in  $M(h)$ . If this step cannot be applied anymore, then we arrive at an assignment  $M_q$  with the desired property (†).

Given  $M_q$ , we reduce the capacities (upper quotas) of each hospital  $h \in \mathcal{H}^*$  by  $\underline{q}(h)$ , set all lower quotas to 0, and delete all residents in  $\mathcal{R}^* := M_q(\mathcal{H}^*)$ . We then find a stable assignment  $M_s$  in the resulting instance  $I'$ ; note that  $I'$  is an instance of HR, so we can find  $M_s$  in polynomial time [8]. Finally, we output  $M^{\text{out}} = M_s \cup M_q$ . Clearly,  $M^{\text{out}}$  is feasible. Also, any blocking pair that  $M^{\text{out}}$  admits must involve either a hospital from  $\mathcal{H}^*$  or a resident from  $\mathcal{R}^* = M_q(\mathcal{H}^*)$  by the stability of  $M_s$  with respect to  $I'$ . Observe that if some  $h \in \mathcal{H}^*$  is involved in some blocking pair  $\{r, h\}$  of  $M^{\text{out}}$ , then we must have  $r \in \mathcal{R}^*$ . To see this, recall that each resident that is preferred by  $h$  to its least preferred resident in  $M_q(h)$  must be in  $\mathcal{R}^*$  because of property (†), and furthermore,  $h$  is under-subscribed in  $M^{\text{out}}$  (within  $I$ ) if and only if  $h$  is under-subscribed in  $M_s$  (within  $I'$ ). Therefore, we can conclude that each blocking pair for  $M^{\text{out}}$  must involve some resident in  $\mathcal{R}^*$ ; recall  $|\mathcal{R}^*| \leq \underline{q}(h)$ . Since each resident in  $\mathcal{R}^*$  is incident to at most  $\Delta_{\mathcal{R}} - 1$  edges not in  $M^{\text{out}}$ , we also have that  $M^{\text{out}}$  admits at most  $(\Delta_{\mathcal{R}} - 1)|\mathcal{R}^*| \leq (\Delta_{\mathcal{R}} - 1)\underline{q}(h)$  blocking pairs.  $\square$

Clearly, the algorithm of Theorem 5 can be used to give a  $(\Delta_{\mathcal{R}} - 1)\underline{q}_{\Sigma}$ -factor approximation as follows. First, we find a stable assignment  $M_s$  for  $I$  in polynomial time using the extension of the Gale-Shapley algorithm for the HOSPITALS/RESIDENTS (HR) problem. If  $M_s$  is not feasible, then by the Rural Hospitals Theorem [9], we know that any feasible assignment for  $I$  must admit at least one blocking pair; hence, the algorithm presented in Theorem 5 clearly yields an approximation with (multiplicative and also additive) factor  $(\Delta_{\mathcal{R}} - 1)\underline{q}_{\Sigma}$ .

Additionally, we also state an analog of Theorem 5 that deals with SMC: it can handle covering constraints on both sides, but assumes that all capacities are 1.

**Theorem 6.** *There is an algorithm that in polynomial time either outputs a feasible assignment for an instance  $I$  of SMC with at most  $(\Delta_{\mathcal{W}} - 1)|\mathcal{M}^*| + (\Delta_{\mathcal{M}} - 1)|\mathcal{W}^*|$  blocking pairs, or concludes that no feasible matching exists for  $I$ .*

*Proof.* We start by finding an arbitrary matching  $M$  that covers each distinguished person (if no such matching exists, then we can stop and reject); such a matching, if existent, can be found in polynomial time by standard flow techniques. Let  $\mathcal{U}^*$  be the set of those persons  $x \in \mathcal{X}^*$  whose partner  $M(x)$  is also in  $\mathcal{X}^*$ .

We proceed by modifying  $M$  into a matching  $M_q$  that covers  $\mathcal{X}^*$  and has the following property:

*If a person  $x \in \mathcal{X}^* \setminus \mathcal{U}^*$  belongs to a blocking pair  $\{x, y\}$  for  $M_q$ , then  $M_q(y) \in \mathcal{X}^*$ .* (✕)

Such an assignment can be obtained as follows. We greedily re-assign partners to the men and women in  $\mathcal{X}^* \setminus \mathcal{U}^*$ , one-by-one: at each step, we take a person  $x \in \mathcal{X}^* \setminus \mathcal{U}^*$ , and if  $x$  forms a blocking pair (with respect to the current matching) with some  $y$  that is not the partner of a distinguished person, then we replace the partner of  $x$  with  $y$ : we add the edge  $\{x, y\}$  to the matching, and delete all other edges incident to  $x$  or  $y$ . Observe that the obtained matching is still feasible. If this step cannot be applied anymore, then we arrive at a matching  $M_q$  with the desired property (✕). We assume, without loss of generality, that each edge in  $M_q$  is incident to some distinguished person, (since all other edges can simply be omitted).

Given  $M_q$ , we delete all men and women covered by  $M_q$ . We then find a stable matching  $M_s$  in the resulting instance  $I'$ ; note that  $I'$  is an instance of STABLE MARRIAGE, so we can find  $M_s$  in polynomial

time [8]. Finally, we output  $M^{\text{out}} = M_s \cup M_q$ . Clearly,  $M^{\text{out}}$  is feasible. Also, any blocking pair that  $M^{\text{out}}$  admits must involve a person covered by  $M_q$  by the stability of  $M_s$  with respect to  $I'$ .

Observe that any blocking pair  $\{x, y\}$  involves a person whose partner by  $M_q$  is distinguished, so either  $M_q(x) \in \mathcal{X}^*$  or  $M_q(y) \in \mathcal{X}^*$ . To see this, first note that if  $x$  is not distinguished, then  $M_q(x)$  must be distinguished, because each edge of  $M_q$  contains a distinguished person. Second, if  $x \in \mathcal{X}^*$ , then either  $x \in \mathcal{U}^*$  (in which case  $M_q(x) \in \mathcal{X}^*$ ) or  $M_q(y) \in \mathcal{X}^*$  because of property  $(\spadesuit)$ . Therefore, we can conclude that each blocking pair for  $M^{\text{out}}$  must involve the partner of some distinguished resident. The partners of distinguished women can be incident to at most  $|\mathcal{W}^*|(\Delta_{\mathcal{M}} - 1)$  blocking pairs, and similarly, the partners of distinguished men can be incident to at most  $|\mathcal{M}^*|(\Delta_{\mathcal{W}} - 1)$  blocking pairs, proving the theorem.  $\square$

## 5 SMC with Bounded Number of Distinguished Persons or Blocking Pairs

In Theorem 3 we proved  $W[1]$ -hardness of SMC-1 for the case where  $\Delta_{\mathcal{M}} = \Delta_{\mathcal{W}} = 3$ , with parameter  $b + |\mathcal{W}^*|$ . Here we investigate those instances of SMC and SMC-1 where preference lists may be unbounded, but either  $b$ , or the number of distinguished persons is constant.

First, if the number  $b$  of blocking pairs allowed is constant, then SMC can be solved by simply running the Gale-Shapley algorithm after guessing and deleting all blocking pairs. This complements the polynomial-time algorithm for SMC-1 by Hamada et al. [12].

**Observation 1.** *SMC can be solved in time  $O(|I|^{b+1})$ , where  $b$  denotes the number of blocking pairs allowed in the input instance  $I$ .*

In Theorem 7 we prove hardness of SMC-1 even if only one woman must be covered. If we require preferences to follow master lists, then a slightly weaker version of Theorem 7, where  $|\mathcal{W}^*| = 2$ , still holds.

**Theorem 7.** *SMC-1 is  $W[1]$ -hard parameterized by  $b + \Delta_{\mathcal{M}}$ , even if  $\mathcal{W}^* = \{s\}$ ,  $\Delta_{\mathcal{W}} \leq 3$ , and  $|L(s)| = 1$ .*

*Proof.* We present a reduction based on the one from MULTICOLORED CLIQUE given in the proof of Theorem 3. Given some graph  $G$  and an integer  $k$  as inputs, we are going to re-use the instance  $I$  constructed in the proof of Theorem 3. Recall that  $I$  has a feasible matching with at most  $b = \binom{k}{2} + 2k$  blocking pairs exactly if  $G$  has a clique of size  $k$ . Recall also that the set of women that must be covered in  $I$  is  $S \cup T \cup U$ ; here we denote this set by  $\mathcal{W}_I^*$ .

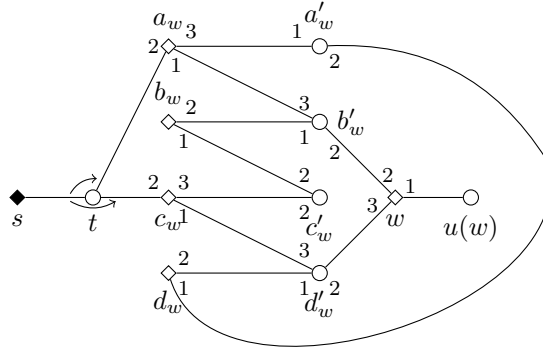


Figure 3: Illustration depicting the forcing gadget  $F_w$  in the proof of Theorem 7.

We define a modified instance  $I'$  of SMC as follows. For each  $w \in \mathcal{W}_I^*$ , we create a *forcing gadget*  $F_w$  containing the newly introduced women  $a_w, b_w, c_w, d_w$  and men  $a'_w, b'_w, c'_w, d'_w$ . We also add the distinguished woman  $s$ , who must be covered in  $I'$ , and the unique man  $t$  in  $L(s)$ . See Fig. 3 for an illustration.

Let  $n(w)$  denote the unique man acceptable for some  $w \in \mathcal{W}_I^*$  in  $I$ . Further, let  $Y = \{a_w, c_w \mid w \in \mathcal{W}_I^*\}$ , and we write  $[Y]$  for an arbitrarily fixed ordering of the elements of  $Y$ . The preferences of the newly introduced men and women, as well as the modified preferences of those agents that find them acceptable, is given below. Here, again, indices take all possible values, and  $w$  can be any woman in  $\mathcal{W}_I^*$ . We let  $I'$  contain all other women and men defined in  $I$ , having the same preferences as in  $I$ .

$$\begin{aligned} L(s) &= (t), & L(t) &= ([Y], s), \\ L(a_w) &= (b'_w, t, a'_w), & L(a'_w) &= (a_w, d_w), \\ L(b_w) &= (c'_w, b'_w), & L(b'_w) &= (b_w, w, a_w), \\ L(c_w) &= (d'_w, t, c'_w), & L(c'_w) &= (c_w, b_w), \\ L(d_w) &= (a'_w, d'_w), & L(d'_w) &= (d_w, w, c_w), \\ L(w) &= (n(w), b'_w, d'_w). \end{aligned}$$

We will show that  $I'$  has a feasible matching with at most  $b$  blocking pairs if and only if  $I$  has such a matching; this clearly proves the theorem.

First observe that any feasible matching  $M'$  for  $I'$  contains the edge  $\{s, t\}$ . Thus, if some woman  $y$  in  $Y$  is not matched by  $M'$  to her first choice, then  $\{y, t\}$  is blocking in  $M'$ . Consider now  $F_w$  for some  $w \in \mathcal{W}_I^*$ . It is straightforward to check that if  $M'(w) \neq n(w)$ , then there are at least two blocking pairs incident to a woman in  $F_w$ . Indeed, assume first that  $\{t, a_w\}$  is the only blocking pair in  $F_w$ ; this quickly implies  $M'(c_w) = d'_w$  and  $M'(b'_w) = w$ , which in turn leads to  $\{a_w, b'_w\}$  blocking  $M'$ , a contradiction. Second, assume that  $\{t, a_w\}$  does not block  $M'$ ; from this follows  $M'(a_w) = b'_w$  and we have that  $\{b'_w, w\}$  is a blocking pair for  $M'$ . Now either  $\{t, c_w\}$  is blocking (in which case our claim holds), or we get  $M'(c_w) = d'_w$ , which implies that  $\{d'_w, s\}$  blocks  $M'$ , again a contradiction.

Now, let  $\mathcal{W}_i$  be the women in  $G_i$  that must be covered in  $I$ , i.e.,  $\mathcal{W}_i = \{s_i, t_i, u_i^1, \dots, u_i^{b+1}\}$ . Consider the number  $\beta_i$  of blocking pairs for  $M'$  that involve a woman either in the gadget  $G_i$  or in a gadget  $F_w$  for some  $w \in \mathcal{W}_i$ . On the one hand, if some  $w \in \mathcal{W}_i$  is not matched by  $M'$  to  $n(w)$ , then  $\beta_i \geq 2$  because of the blocking pairs in  $F_w$ . On the other hand, if each  $w \in \mathcal{W}_i$  is matched by  $M'$  to  $n(w)$ , then using the arguments of the proof for Theorem 3, we again know  $\beta_i \geq 2$  because of the blocking pairs in  $G_i$ . Also,  $\beta_i = 2$  can only be achieved if (i)  $M'(t_i) = n(t_i)$ , as otherwise  $\{t_i, n(t_i)\}$  would be blocking for  $M'$ , in addition to the two blocking pairs in  $F_{t_i}$ , and (ii)  $M'(u_i^h) = n(u_i^h)$  for each  $h \in \{1, \dots, b+1\}$ , as otherwise we would have  $M'(s_i) = n(s_i)$  (so as to avoid having four blocking pairs due to women in  $F_{u_i^h}$  and  $F_{s_i}$ ), implying at least one blocking pair in  $G_i$  in addition to those in  $F_{u_i^h}$ .

Analogously, let  $\beta_{i,j}$  denote the number of blocking pairs for  $M'$  that involve a woman either in the gadget  $G_{\{i,j\}}$  or in a gadget  $F_w$  for some  $w \in \{s_{i,j}, t_{i,j}\}$ . Then either  $\beta_{i,j} \geq 2$ , or we know that  $M'(w) = n(w)$  for both women  $w \in \{s_{i,j}, t_{i,j}\}$ ; in this case, from the proof of Theorem 3 we get  $\beta_{i,j} \geq 1$ . However, supposing that  $M'$  has at most  $b = 2k + \binom{k}{2}$  blocking pairs, it follows that  $\beta_i = 2$  and  $\beta_{i,j} = 1$  must hold for each  $i \in \{1, \dots, k\}$  and each  $i, j$  with  $1 \leq i < j \leq k$ , respectively.

Along the same lines as in the proof of Theorem 3, it can also be verified that  $\beta_{i,j} = 1$  for each pair of indices  $i, j$  can only be achieved if  $M' \triangle M_s$  contains a path in each gadget  $G_i$ . From  $M'(w) = n(w)$  for each  $w \in \mathcal{W}_i \setminus \{s_i\}$  we get that such a path contains at least one blocking pair. This implies  $M'(s_i) = n(s_i)$ , as otherwise we would end up with  $\beta_i \geq 3$  because of the blocking pairs incident to women of  $F_{s_i}$ .

Altogether, we have proved that  $M'(w) = n(w)$  for each  $w \in \mathcal{W}_I^*$ . Hence, the restriction of  $M'$  to  $I$  yields a feasible matching for  $I$  that admits at most  $b$  blocking pairs.

For the other direction, suppose that  $I$  has a feasible matching  $M$ . Then it is easy to see that adding the edges  $\{a_w, b'_w\}$ ,  $\{b_w, c'_w\}$ ,  $\{c_w, d'_w\}$ , and  $\{d_w, a'_w\}$  for each  $w \in \mathcal{W}_I^*$  together with the edge  $\{s, t\}$  to  $M$  yields a feasible matching for  $I'$  that contains exactly the same number of blocking pairs in  $I'$  as  $M$  does in  $I$ .  $\square$

**Theorem 8.** *SMC-1 is  $W[1]$ -hard parameterized by  $b + \Delta_{\mathcal{M}}$ , even if there is a master list over men as well as one over women,  $|\mathcal{W}^*| = 2$ ,  $\Delta_{\mathcal{W}} \leq 3$ , and  $|L(w)| = 1$  for each  $w \in \mathcal{W}^*$ .*

*Proof.* The proof is very similar to the one for Theorem 7, so we will only sketch it. Again, we are going to re-use the instance  $I$  constructed in the proof of Theorem 3, and construct a modified instance  $I'$  of SMC, adding only two new women  $z_1$  and  $z_2$  and two men  $m_1$  and  $m_2$  to  $I$ . We append  $z_1$  and  $z_2$ , in this order,

to the master list over women, and similarly, we append  $m_1, m_2$  to the master list over men. We define the women to be covered in  $I'$  as  $z_1$  and  $z_2$ .

Again, we denote the set of women to be covered in  $I$  by  $\mathcal{W}_I^*$ , and we denote by  $n(w)$  the unique man acceptable for some  $w \in \mathcal{W}_I^*$  in  $I$ . The preferences of the newly introduced men and women, as well as the modified preferences of those agents that find them acceptable, is given below (here,  $[\mathcal{W}_I^*]_{\prec}$  denotes the ordering of  $\mathcal{W}_I^*$  given by the master list). We let  $I'$  contain all other women and men defined in  $I$ , having the same preferences as in  $I$ .

$$\begin{aligned} L(z_1) &= (m_1), & L(m_1) &= ([\mathcal{W}_I^*]_{\prec}, z_1), \\ L(z_2) &= (m_2), & L(m_2) &= ([\mathcal{W}_I^*]_{\prec}, z_2), \\ L(w) &= (n(w), m_1, m_2) & \forall w \in \mathcal{W}_I^*. \end{aligned}$$

Arguing analogously as before in the proof of Theorem 7, one can show that  $I'$  has a feasible matching with at most  $b$  blocking pairs if and only if  $I$  has such a matching; this suffices to prove the theorem.  $\square$

To contrast these intractability results for SMC-1, we propose an algorithm which runs in polynomial time if not only  $|\mathcal{W}^*|$  but also  $\Delta_{\mathcal{M}}$  is constant. The algorithm relies on the observation that in this case, the number of blocking pairs in an optimal solution is at most  $|\mathcal{W}^*|(\Delta_{\mathcal{M}} - 1)$  by Theorem 5.

**Theorem 9.** *SMC-1 instances can be solved in time  $O(b_{\max}^2(|\mathcal{W}|(|\mathcal{M}| + b_{\max}))^{b_{\max}+1})$ , where  $b_{\max} = |\mathcal{W}^*|(\Delta_{\mathcal{M}} - 1)$ .*

*Proof.* By Theorem 5, there is a matching with at most  $b_{\max} = |\mathcal{W}^*|(\Delta_{\mathcal{M}} - 1)$  blocking pairs. Hence, if the number  $b$  of blocking pairs allowed is at least  $b_{\max}$ , then we can simply run the algorithm of Theorem 5. Otherwise, we can use the algorithm of Hamada et al. [12], which gives us the required run time.  $\square$

To deal with covering constraints on both sides, we can use Theorem 6 instead of Theorem 5. This yields that if both the number of distinguished persons and the maximum length of all preference lists is constant, then SMC becomes polynomial-time solvable.

**Theorem 10.** *SMC can be solved in time  $O(|I|^{b_{\max}+1})$ , for  $b_{\max} = (\Delta_{\mathcal{M}} - 1)|\mathcal{W}^*| + (\Delta_{\mathcal{W}} - 1)|\mathcal{M}^*|$ .*

*Proof.* By Theorem 6, there is a matching with at most  $(\Delta_{\mathcal{M}} - 1)|\mathcal{W}^*| + (\Delta_{\mathcal{W}} - 1)|\mathcal{M}^*|$  blocking pairs. Hence, if the number  $b$  of blocking pairs allowed is at least  $b_{\max}$ , then we can simply run the algorithm of Theorem 6. Otherwise, we can use Observation 1, which gives us the required run time.  $\square$

The above theorem shows that if each of  $|\mathcal{W}^*|$ ,  $|\mathcal{M}^*|$ ,  $\Delta_{\mathcal{W}}$ , and  $\Delta_{\mathcal{M}}$  is constant, then the problem becomes polynomial-time solvable. By contrast, if  $|\mathcal{W}^*| \geq 1$  and  $|\mathcal{M}^*| \geq 1$ , then Theorem 7 implies immediately that restricting the maximum length of the preference lists on only one side still results in a hard problem: if  $\Delta_{\mathcal{W}} \leq 3$ , then SMC-1 (and hence SMC) still remains W[1]-hard with parameter  $b + \Delta_{\mathcal{M}}$ , even if  $|\mathcal{W}^*| = 1$  (and, in the context of SMC,  $|\mathcal{M}^*| = 0$ ). On the other hand, Theorem 3 shows that the problem remains hard even if all preference lists have length at most 3 and  $|\mathcal{M}^*| = 0$ . Therefore, restricting only three of the values  $|\mathcal{W}^*|$ ,  $|\mathcal{M}^*|$ ,  $\Delta_{\mathcal{W}}$ , and  $\Delta_{\mathcal{M}}$  to be constant does not yield tractability for SMC. Hence, Theorem 10 is tight in this sense.

## 6 SMC with Preference Lists of Length at most Two

In this section we investigate the computational complexity of SMC where the maximum length of preference lists is bounded by 2 on one side. This restriction leads to important tractable special cases: we obtain both polynomial-time algorithms and fixed-parameter tractability results for various parameterizations.

Let  $I$  be an instance of SMC, with underlying graph  $G$ . Let  $M_s$  be a stable matching in  $I$ , and let  $\mathcal{M}_0^*$  and  $\mathcal{W}_0^*$  denote the set of distinguished men and women, respectively, unmatched by  $M_s$ . Furthermore, let  $\mathcal{M}_0$  and  $\mathcal{W}_0$  denote the set of all men and women, respectively, unmatched by  $M_s$ . An  $M_s$ -alternating path (that is, a path that alternates between edges of  $M_s$  and edges not in  $M_s$ ) in  $G$  is called an *augmenting*

path, if it contains a distinguished person not covered by  $M_s$ . We will call an augmenting path  $P$  *masculine* or *feminine*, if it contains a man in  $\mathcal{M}_0^*$  or a woman in  $\mathcal{W}_0^*$ ; if  $P$  is both masculine and feminine, then we call it *neutral*. If  $P$  is not neutral, then we say that it *starts* at the (unique) distinguished person it contains, and *ends* at its other endpoint.

## 6.1 Covering constraints on one side

Here we deal with the SMC-1 problem where only women need to be covered. We first give a polynomial-time algorithm for SMC-1 when each man finds at most two women acceptable, and then show NP-hardness of SMC-1 for instances where each woman finds at most two men acceptable. We start by considering the special case of SMC-1 where  $\Delta_{\mathcal{M}} \leq 2$ .

**Theorem 11.** *There is a polynomial-time algorithm for the special case of SMC-1 where each man finds at most two women acceptable.*

We now present the algorithm that proves Theorem 11. It is easy to prove the following properties of augmenting paths using that  $\Delta_{\mathcal{M}} \leq 2$ :

**Proposition 1.** *For augmenting paths  $P_1$  and  $P_2$  starting at  $w_1$  and  $w_2$ , resp., we have:*

- (a) *If  $w_1 \neq w_2$ , then  $P_1$  and  $P_2$  are either vertex-disjoint, or they both end at some  $m \in \mathcal{M}_0$ , with  $V(P_1) \cap V(P_2) = \{m\}$ .*
- (b) *If there is an edge  $\{m, w\}$  of  $G$  (with  $m \in \mathcal{M}$  and  $w \in \mathcal{W}$ ) connecting  $P_1$  and  $P_2$ , then  $m \in \mathcal{M}_0$  and  $m$  must be the endpoint of  $P_1$  or  $P_2$ .*
- (c) *If  $w_1 = w_2$  and  $P$  is the common subpath of  $P_1$  and  $P_2$ , then either  $V(P_1) \cap V(P_2) = V(P)$ , or  $P_1$  and  $P_2$  both end at some  $m \in \mathcal{M}_0$  and  $V(P_1) \cap V(P_2) = V(P) \cup \{m\}$ .*

With an augmenting path (or, more generally, with a set of edges)  $P$  we associate a *cost*, which is the number of blocking pairs that  $M_s \Delta P$  admits. A pair  $\{m, w\}$  for some  $m \in \mathcal{M}$  and  $w \in \mathcal{W}$  is *special*, if  $m \in \mathcal{M}_0$  and  $w$  is the second (less preferred) woman in  $L(m)$ . As it turns out, such edges can be ignored during certain steps of the algorithm; thus, we define the *special cost* of  $P$  as the number of non-special blocking pairs in  $M_s \Delta P$ .

**Lemma 12.** *For vertex-disjoint augmenting paths  $P_1$  and  $P_2$  with costs  $c_1$  and  $c_2$ , resp., the cost of  $P_1 \cup P_2$  is at most  $c_1 + c_2$ . Further, if the cost of  $P_1 \cup P_2$  is less than  $c_1 + c_2$ , then the following holds for  $\{i_1, i_2\} = \{1, 2\}$ : there is a special edge  $\{m, w\}$  with  $P_{i_1}$  ending at  $m$  and  $w$  appearing on  $P_{i_2}$ ; moreover,  $\{m, w\}$  is blocking in  $M_s \Delta P_{i_2}$ , but not in  $M_s \Delta (P_1 \cup P_2)$ .*

*Proof.* First observe that if some edge  $\{m, w\}$  has a common vertex with only one of the paths  $P_1$  and  $P_2$ , say  $P_1$ , then  $\{m, w\}$  is blocking in  $M_s \Delta P_1$  if and only if it is blocking in  $M_s \Delta (P_1 \cup P_2)$ .

Consider now the case when  $\{m, w\}$  connects  $P_1$  and  $P_2$ . By Proposition 1, this implies that one of the paths, say  $P_1$ , ends at  $m \in \mathcal{M}_0$  (and  $w$  lies on  $P_2$ ). Clearly,  $\{m, w\}$  is not blocking in  $M_s \Delta P_1$ , by the stability of  $M_s$ . If, on the one hand,  $w$  is the first choice of  $m$ , then  $\{m, w\}$  is blocking in  $M_s \Delta P_2$  exactly if it is blocking in  $M_s \Delta (P_1 \cup P_2)$ . If, on the other hand,  $\{m, w\}$  is special, then it cannot be blocking in  $M_s \Delta (P_1 \cup P_2)$ , but it might be blocking in  $M_s \Delta P_2$ . Putting all these facts together, the lemma follows immediately.  $\square$

We are ready to provide the algorithm, in a sequence of four steps.

### Step 1: Computing all augmenting paths.

By Proposition 1, if we delete  $\mathcal{M}_0$  from the union of augmenting paths starting at some  $w \in \mathcal{W}_0^*$ , then we obtain a tree. Furthermore, these trees are mutually vertex-disjoint for different starting vertices of  $\mathcal{W}_0^*$ . This allows us to compute all augmenting paths in linear time, e.g., by an appropriately modified version of the

DFS algorithm (so that only augmenting paths are considered). During this process, we can also compute the special cost of each augmenting path in a straightforward way.

**Step 2: Constructing an auxiliary graph.**

Using the results of the computation of Step 1, we construct an edge-weighted single bipartite graph  $G_{\text{path}}$  as follows. The vertex set of  $G_{\text{path}}$  is the union of  $\mathcal{W}_0^*$  and  $\mathcal{M}_0 \cup \{w' \mid w \in \mathcal{W}_0^*\}$ , where for each woman  $w \in \mathcal{W}_0^*$  we create a new corresponding vertex  $w'$ . We add an edge between  $w \in \mathcal{W}_0^*$  and  $m \in \mathcal{M}_0$  with weight  $c$  if there exists an augmenting path with endpoints  $w$  and  $m$  having special cost  $c$ . Further, for each  $w \in \mathcal{W}_0^*$  we compute the minimum special cost  $c_w^{\min}$  of any augmenting path starting at  $w$  and not ending in  $\mathcal{M}_0$ , and add an edge between  $w$  and  $w'$  with weight  $c_w^{\min}$  in  $G_{\text{path}}$ .

**Step 3: Computing a minimum weight matching.**

We compute a matching  $M_P$  in  $G_{\text{path}}$  covering  $\mathcal{W}_0^*$  and having minimum weight. Observe that such a matching corresponds to a set of augmenting paths  $\mathcal{P} = \{P_w \mid w \in \mathcal{W}_0^*\}$  that are mutually vertex-disjoint by Proposition 1. Recall that the special cost of  $P_w$  is the weight of the edge in  $M_P$  incident to  $w$ .

**Step 4: Eliminating blocking special edges.**

In this step, we modify  $\mathcal{P}$  iteratively. We start by setting  $\mathcal{P}_{\text{act}} = \mathcal{P}$ . At each iteration we modify  $\mathcal{P}_{\text{act}}$  as follows. We check whether there exists a special edge  $\{m^*, w^*\}$  that is blocking in  $M_s \triangle \mathcal{P}_{\text{act}}$ . If yes, then notice that  $m^*$  is not matched in  $M_s \triangle \mathcal{P}_{\text{act}}$ , because  $\{m^*, w^*\}$  is special. Let  $P$  be the path of  $\mathcal{P}_{\text{act}}$  containing  $w^*$ . We modify  $\mathcal{P}_{\text{act}}$  by truncating  $P$  to its subpath between its starting vertex and  $w^*$ , and appending to it the edge  $\{m^*, w^*\}$ . This way,  $\{m^*, w^*\}$  becomes an edge of the matching  $M_s \triangle \mathcal{P}_{\text{act}}$ . The iteration stops when there is no special edge blocking  $M_s \triangle \mathcal{P}_{\text{act}}$ . Note that once a special edge ceases to be blocking in  $M_s \triangle \mathcal{P}_{\text{act}}$ , it cannot become blocking again during this process, so the algorithm performs at most  $|\mathcal{M}_0|$  iterations. For each  $w \in \mathcal{W}_0^*$ , let  $P_w^*$  denote the augmenting path in  $\mathcal{P}_{\text{act}}$  covering  $w$  at the end of Step 4; we define  $\mathcal{P}^* = \{P_w^* \mid w \in \mathcal{W}_0^*\}$  and output the matching  $M_s \triangle \mathcal{P}^*$ .

This completes the description of the algorithm; we now provide its analysis.

**Lemma 13.**  $M_{\text{sol}} := M_s \triangle \mathcal{P}^*$  is a feasible matching for  $I$ , and the number of blocking pairs for  $M_{\text{sol}}$  is at most the weight of  $M_P$ .

*Proof.* Let us consider the situation when the iteration in Step 4 deals with a special edge  $\{m^*, w^*\}$  blocking in  $\mathcal{P}_{\text{act}}$ . Notice that since  $w^*$  is the second woman in  $L(m^*)$  (by the definition of a special edge), and since  $\{w^*, m^*\}$  is blocking in  $M_s \triangle \mathcal{P}_{\text{act}}$ , we know that  $m^*$  is unmatched in  $M_s \triangle \mathcal{P}_{\text{act}}$ , that is,  $m^*$  does not lie on any of the augmenting paths in  $\mathcal{P}_{\text{act}}$ . From this follows that the augmenting paths in  $\mathcal{P}_{\text{act}}$ , and hence in  $\mathcal{P}^*$ , remain mutually vertex-disjoint. Therefore,  $M_{\text{sol}}$  is indeed a matching, and as it covers  $\mathcal{W}_0^*$ , it is feasible.

Clearly, Step 4 ensures that there are no blocking special edges in  $M_{\text{sol}}$ . Note that when the algorithm modifies  $P_w$  for some  $w \in \mathcal{W}_0^*$ , at most one new blocking pair may arise with respect to  $M_s \triangle \mathcal{P}_{\text{act}}$ , and from the stability of  $M$  and Proposition 1 it follows that such an edge must be a special edge (incident to the man at which  $P_w$  ends before its modification). Hence, we obtain that the cost of  $P_w^*$  is at most the special cost of  $P_w$ , for each  $w \in \mathcal{W}_0^*$ . By Lemma 12, the number of blocking pairs that  $M_{\text{sol}}$  admits is at most the sum of the costs of all augmenting paths in  $\mathcal{P}^*$ ; this finishes the proof.  $\square$

To show that our algorithm is correct, by Lemma 13 it remains to prove that there exists a matching covering  $\mathcal{W}_0^*$  in  $G_{\text{path}}$  with weight at most the number of blocking pairs in  $M^{\text{opt}}$ , where  $M^{\text{opt}}$  denotes an optimal solution in  $I$ . Clearly,  $M_s \triangle M^{\text{opt}}$  contains an augmenting path  $Q_w$  covering  $w$  for each  $w \in \mathcal{W}_0^*$ . If some  $Q_w$  ends at a man  $m \in \mathcal{M}_0$ , then clearly no other path in  $M_s \triangle M^{\text{opt}}$  can end at  $m$ . So let us take the matching  $M_Q$  in  $G_{\text{path}}$  that includes all pairs  $\{m, w\}$  where  $Q_w$  ends at  $m \in \mathcal{M}_0$  for some  $w \in \mathcal{W}_0^*$ . Also, we put  $\{w, w'\}$  into  $M_Q$  if  $Q_w$  does not end at a man of  $\mathcal{M}_0$ . Note that  $M_Q$  is indeed a matching.

It remains to show that the weight of  $M_Q$  is at most the number of blocking pairs in  $M^{\text{opt}}$ . By definition, the weight of  $M_Q$  is at most the sum of the special costs of the paths  $Q_w$  for every  $w \in \mathcal{W}_0^*$ . By Lemma 12, any non-special blocking pair in  $M_s \triangle Q_w$  remains a blocking pair in  $M_s \triangle (\bigcup_{w \in \mathcal{W}_0^*} Q_w)$ , and hence in  $M^{\text{opt}}$  as well. Hence, there is a matching in  $G_{\text{path}}$  with weight at most the number of blocking pairs in an optimal

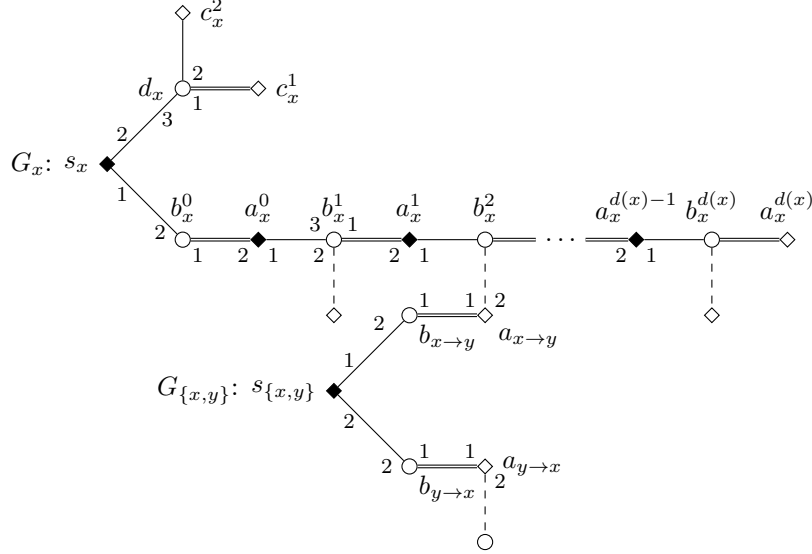


Figure 4: Illustration of a node gadget  $G_x$  and an edge gadget  $G_{\{x,y\}}$  constructed in the proof of Theorem 14. Double edges denote edges of a stable matching for  $I$ , and dashed edges are those leaving some gadget. The example depicted assumes  $y = n(x, 2)$ .

solution, implying the correctness of our algorithm. As the algorithm runs in polynomial time, Theorem 11 follows.

By contrast to Sect. 6.1, if men may have preference lists of length 3, then SMC-1 (and hence SMC) is NP-hard even if each woman finds at most two men acceptable.

**Theorem 14.** *SMC-1 is NP-hard even if women's preference lists have length at most 2 and men's preference lists have length at most 3.*

*Proof.* We give a reduction from the NP-hard VERTEX COVER problem, asking whether the input graph  $G$  has a vertex cover of size at most  $k$ . We order the vertices of  $G$  arbitrarily, and denote the  $h$ -th neighbor of some vertex  $x$  by  $n(x, h)$  for any  $h \in \{1, \dots, d(x)\}$ .

Let us construct an instance  $I$  of SMC as follows; see Fig. 4 for an illustration. For each vertex  $x \in V(G)$  we construct a *node gadget*  $G_x$  which contains women  $s_x, a_x^0, \dots, a_x^{d(x)}, c_x^1$  and  $c_x^2$ , and men  $b_x^0, \dots, b_x^{d(x)}$ , and  $d_x$ . For each edge  $\{x, y\} \in E(G)$  we also construct an *edge gadget*  $G_{\{x,y\}}$  involving women  $s_{\{x,y\}}, a_{x \rightarrow y}$  and  $a_{y \rightarrow x}$ , and men  $b_{x \rightarrow y}$  and  $b_{y \rightarrow x}$ . Furthermore, there are two edges in the underlying graph connecting  $G_{\{x,y\}}$  to  $G_x$  and  $G_y$ , namely  $\{a_{x \rightarrow y}, b_x^h\}$  and  $\{a_{y \rightarrow x}, b_y^\ell\}$  where  $y = n(x, h)$  and  $x = n(y, \ell)$ .

The preference lists of  $I$  are given in Table 4. We define the set of women with covering constraints as

$$\mathcal{W}^* = \{s_x \mid x \in V(G)\} \cup \{s_{\{x,y\}} \mid \{x,y\} \in E(G)\} \cup \{a_x^h \mid 0 \leq h < d(x)\},$$

and set the number of blocking pairs allowed to be  $|V(G)| + k$ .

We are going to prove that  $I$  admits a feasible matching with at most  $|V(G)| + k$  blocking pairs if and only if there is a vertex cover of size  $k$  in the graph  $G$ .

" $\Rightarrow$ ": Let  $M$  be a feasible matching with at most  $|V(G)| + k$  blocking pairs. We say that the *cost* of some gadget  $G_x$  (or  $G_{\{x,y\}}$ ) is the number of edges blocking  $M$  which are incident to some man of  $G_x$  (or  $G_{\{x,y\}}$ , respectively.) We will prove that the set  $S$  of vertices  $x$  for which  $G_x$  has cost at least 2 is a vertex cover of  $G$ .

First, let us consider some  $x$  for which  $M(s_x) = d_x$ . In this case, both  $c_x^1$  and  $c_x^2$  form a blocking pair for  $M$  with  $d_x$ , implying  $x \in S$ . Second, let us consider some  $x$  with  $M(s_x) = b_x^0$ . Since each  $a_x^h$  with



Table 4: Preference lists of women and men in the proof of Theorem 14. When not stated otherwise, indices take all possible values.

$L(s_x)$	$= (b_x^0, d_x),$	
$L(b_x^0)$	$= (a_x^0, s_x),$	
$L(b_x^h)$	$= (a_x^h, a_{x \rightarrow y}, a_x^{h-1})$	where $1 \leq h \leq d(x)$ and $y = n(x, h)$ ,
$L(a_x^h)$	$= (b_x^{h+1}, b_x^h)$	where $0 \leq h < d(x)$ ,
$L(a_x^{d(x)})$	$= (b_x^{d(x)}),$	
$L(c_x^h)$	$= (d_x),$	
$L(d_x)$	$= (c_x^1, c_x^2, s_x),$	
$L(s_{\{x,y\}})$	$= (b_{x \rightarrow y}, b_{y \rightarrow x})$	where $x$ precedes $y$ ,
$L(b_{x \rightarrow y})$	$= (a_{x \rightarrow y}, s_{\{x,y\}}),$	
$L(a_{x \rightarrow y})$	$= (b_{x \rightarrow y}, b_x^h)$	where $y = n(x, h)$ .

$h \in \{0, \dots, d(x) - 1\}$  must be matched by  $M$ , we obtain  $M(a_x^h) = b_x^{h+1}$  for each such  $h$ . Hence,  $a_x^{d(x)}$  and  $b_x^{d(x)}$  form a blocking pair for  $M$ . Moreover, if the woman  $a_{x \rightarrow y}$  is unmatched in  $M$  for some  $y$ , then  $\{a_{x \rightarrow y}, b_x^h\}$  is also a blocking pair in  $M$  (where  $y = n(x, h)$ ), and implies a cost of at least 2 for  $G_x$ . Therefore, we can observe that if  $x \notin S$ , then  $a_{x \rightarrow y}$  must be matched by  $M$  to  $b_{x \rightarrow y}$  for each neighbor  $y$  of  $x$  in  $G$ .

However, for any  $\{x, y\} \in E(G)$ ,  $M$  must match  $s_{\{x,y\}}$  either to  $b_{x \rightarrow y}$  or to  $b_{y \rightarrow x}$ , which means that  $x \in S$  or  $y \in S$ . This proves that  $S$  is indeed a vertex cover for  $G$ . Moreover, the number of vertices in  $S$  can be at most  $k$ , since each  $G_x$  with  $x \in S$  has cost at least 2, each  $G_x$  with  $x \notin S$  has cost at least 1, and the total cost of all gadgets cannot exceed our budget  $|V(G)| + k$ .

“ $\Leftarrow$ ”: Given a vertex cover  $S$  of size at most  $k$  for  $G$ , we define a matching  $M$  with the desired properties. Namely, for each  $x \in S$  we set  $M(s_x) = d_x$  and  $M(a_x^h) = b_x^h$  for each  $h \in \{0, \dots, d(x)\}$ . In this case,  $c_x^1, c_x^2$  are unmatched by  $M$ , both forming a blocking pair with  $d_x$ . By contrast, all of the men  $b_x^0, \dots, b_x^{d(x)}$  get their first choices.

Next, for each  $x \in V(G) \setminus S$  we set  $M(s_x) = b_x^0$ ,  $M(c_x^1) = d_x$ , and  $M(a_x^h) = b_x^{h+1}$  for each  $h \in \{0, \dots, d(x) - 1\}$ . Note that  $a_x^{d(x)}$  is unmatched by  $M$ , and thus forms a blocking pair with  $b_x^{d(x)}$ . Observe also that  $d_x$  is not contained in any blocking pair.

Finally, for some  $\{x, y\} \in E(G)$ , let us assume  $y \in S$  (since  $S$  is a vertex cover, it contains  $x$  or  $y$ ). We set  $M(s_{\{x,y\}}) = b_{y \rightarrow x}$  and  $M(a_{x \rightarrow y}) = b_{x \rightarrow y}$ . Note that  $a_{x \rightarrow y}$  gets her first choice, so it cannot be involved in a blocking pair. Although  $a_{y \rightarrow x}$  is unmatched by  $M$ , we know that it cannot form a blocking pair with  $b_y^\ell$  where  $x = n(y, \ell)$ , because  $y \in S$  and hence  $b_y^\ell$  is assigned her first choice by  $M$ . Thus, no man or woman of some edge gadget participates in a blocking pair, and therefore we obtain that the total number of blocking pairs for  $M$  is exactly  $|V(G)| + k$ .

Since  $M$  is feasible, the theorem follows.  $\square$

## 6.2 Covering constraints on both sides

Let us now investigate the complexity of SMC with covering constraints both for men and women. If we restrict the maximum length of preference lists on both sides to be at most 2, SMC becomes linear-time solvable: this is easy to see using that if  $\max\{\Delta_{\mathcal{W}}, \Delta_{\mathcal{M}}\} \leq 2$ , then the graph underlying the instance is a collection of paths and cycles.

**Observation 2.** *Instances of SMC with  $\max\{\Delta_{\mathcal{W}}, \Delta_{\mathcal{M}}\} \leq 2$  are polynomial-time solvable.*

Recall that the case where  $\Delta_{\mathcal{W}} = 2$  and  $\Delta_{\mathcal{M}} = 3$  is NP-complete by Theorem 14, even if there are no distinguished men to be covered. However, switching the role of men and women, Theorem 11 shows that if

Table 5: Preference lists of women and men in the proof of Theorem 15. We denote by  $\text{ind}(j, h)$  the index  $i$  for which  $u_i$  is the  $h$ -th element in  $S_j$ . When not stated otherwise, indices take all possible values.

$L(x_0)$	$= (\hat{x}_1),$	
$L(x_j)$	$= (\hat{x}_{j+1}, \hat{x}_j)$	for $1 \leq j < m,$
$L(x_m)$	$= (\hat{x}_m, y),$	
$L(s_j)$	$= (\hat{p}_j^3, \hat{x}_j),$	
$L(p_j^1)$	$= (\hat{p}_j^1, t_j),$	
$L(p_j^h)$	$= (\hat{p}_j^{h-1}, \hat{p}_j^h)$	for $h \in \{2, 3\},$
$L(q_j)$	$= (\hat{q}_j, t_j),$	
$L(a_{i,j})$	$= (\hat{b}_{i,j}, \hat{p}_j^h)$	for $h \in \{1, 2, 3\}$ and $i = \text{ind}(j, h),$
$L(b_{i,j})$	$= (\hat{b}_{i,j}, c_i),$	
$L(\hat{x}_j)$	$= (x_j, s_j, x_{j-1}),$	
$L(y)$	$= (x_m),$	
$L(t_j)$	$= (p_j^1, q_j),$	
$L(\hat{p}_j^h)$	$= (p_j^h, a_{i,j}, p_j^{h+1})$	for $h \in \{1, 2\}$ and $i = \text{ind}(j, h),$
$L(\hat{p}_j^3)$	$= (p_j^3, a_{i,j}, s_j)$	for $h \in \{1, 2\}$ and $i = \text{ind}(j, 3),$
$L(\hat{q}_j)$	$= (q_j),$	
$L(\hat{b}_{i,j})$	$= (b_{i,j}, a_{i,j}),$	
$L(c_i)$	$= ([B_i])$	where $B_i = \{b_{i,j} \mid u_i \in S_j\}$ and $[B_i]$ is some fixed ordering of $B_i$ .

there are no women to be covered, then  $\Delta_{\mathcal{W}} \leq 2$  guarantees polynomial-time solvability for SMC. This raises the natural question whether SMC with  $\Delta_{\mathcal{W}} \leq 2$  can be solved efficiently, if the number of distinguished women is bounded. In the next theorem, we show that this is unlikely, as the problem turns out to be NP-hard.

**Theorem 15.** *SMC is NP-hard, even if  $\Delta_{\mathcal{W}} = 2$ ,  $\Delta_{\mathcal{M}} = 3$  and  $|\mathcal{W}^*| = 1$ .*

*Proof.* We present a reduction from the following special case of EXACT-3-COVER. We are given a set  $U = \{u_1, \dots, u_n\}$ , a family  $\mathcal{S}$  of subsets  $S_1, \dots, S_m$  of  $U$ , each having size 3, such that each element of  $U$  occurs in at most three sets of  $\mathcal{S}$ . The task is to decide whether there exists a collection of  $n/3$  sets in  $\mathcal{S}$  whose union covers  $U$ ; such a collection of subsets is called an *exact cover* for  $U$ . This problem is NP-complete [10]. We construct an equivalent instance  $I$  of SMC as follows.

The set  $\mathcal{W}$  of women in  $I$  contains the women  $s_j$ ,  $p_j^1$ ,  $p_j^2$ ,  $p_j^3$ , and  $q_j$  for each  $j \in \{1, \dots, m\}$ , women  $x_0, x_1, \dots, x_m$ , and two women  $a_{i,j}, b_{i,j}$  for each element  $u_i$  contained in  $S_j$  for each  $j \in \{1, \dots, m\}$ . The men defined in  $I$  are  $\hat{x}_j$ ,  $\hat{p}_j^1$ ,  $\hat{p}_j^2$ ,  $\hat{p}_j^3$ ,  $\hat{q}_j$ , and  $t_j$  for each  $j \in \{1, \dots, m\}$ , a man  $c_i$  for each  $u_i \in U$ , a man  $\hat{b}_{i,j}$  for each element  $u_i$  contained in  $S_j$  for each  $j \in \{1, \dots, m\}$ , plus one additional man  $y$ . (The pairs  $\{w, \hat{w}\}$  form a stable matching in  $I$ .) The only distinguished woman in  $I$  is  $x_0$ , and the set of distinguished men is  $\mathcal{M}^* = \{c_i \mid i = 1, \dots, n\} \cup \{\hat{x}_j, t_j \mid j = 1, \dots, m\} \cup \{y\}$ . The preferences of each person are as shown in Table 5. Note that since each subset  $S_j$  contains three elements, and each element  $u_i$  is contained in at most three subsets from  $\mathcal{S}$ , the constructed instance satisfies  $\Delta_{\mathcal{M}} \leq 3$ . To finish the construction, we set the number of allowed blocking pairs to be  $b = 2m + 2n/3 + 1$ . The construction is illustrated in Fig. 5.

We claim that  $I$  admits a feasible matching with at most  $b$  blocking pairs if and only if  $(U, \mathcal{S})$  is a “yes”-instance of EXACT-3-COVER.

“ $\implies$ ”: Suppose that  $M$  is a feasible matching for  $I$  with at most  $b$  blocking pairs. First, observe that since every  $\hat{x}_j$  for  $j \in \{1, \dots, m\}$ , but also  $x_0$  is distinguished,  $M$  must contain the edges  $\{x_j, \hat{x}_{j+1}\}$  for



exact covering of  $U$ .

“ $\Leftarrow$ ”: Suppose that  $(U, \mathcal{S})$  is a yes-instance of EXACT-3-COVER. Let  $J$  be the set of indices describing a solution, meaning that the subsets  $S_j \in \mathcal{S}$  with  $j \in J$  form an exact covering of  $U$ ; clearly,  $|J| = n/3$ . We define  $\sigma(i)$  as the unique index  $j$  in  $J$  for which  $u_i \in S_j$ . We define a feasible matching  $M$  for  $I$  with exactly  $b$  blocking pairs as follows (indices take all possible values, if not stated otherwise).

$$\begin{aligned} M(\hat{x}_j) &= x_{j-1}, & M(t_j) &= q_j & \text{if } j \notin J, \\ M(y) &= x_m, & M(t_j) &= p_j^1 & \text{if } j \in J, \\ M(c_i) &= b_{i, \sigma(i)}, & M(\hat{p}_j^h) &= p_j^{h+1} & \text{if } j \in J, h \in \{1, 2\}, \\ M(\hat{b}_{i, \sigma(i)}) &= a_{i, \sigma(i)}, & M(\hat{p}_j^3) &= s_j & \text{if } j \in J, \\ M(\hat{w}) &= w & & & \text{if } w \in \mathcal{W} \text{ and neither } M(w) \text{ nor } M(\hat{w}) \text{ is defined yet.} \end{aligned}$$

It is easy to check that  $M$  indeed is feasible, and the blocking pairs it admits are exactly the pairs  $\{x_m, \hat{x}_m\}$ ,  $\{b_{i, \sigma(i)}, \hat{b}_{i, \sigma(i)}\}$  for each  $i \in \{1, \dots, n\}$ ,  $\{p_j^1, \hat{p}_j^1\}$  for each  $j \in J$ ,  $\{q_j^1, \hat{q}_j^1\}$  for each  $j \notin J$ , and  $\{s_j, \hat{x}_j\}$  for each  $j \notin J$ . This proves the theorem.  $\square$

Contrasting Theorem 15, we establish fixed-parameter tractability of the case  $\Delta_{\mathcal{M}} \leq 2$  with two different parameterizations. Considering our five parameters, the relevant cases (whose tractability or intractability does not follow from our results obtained so far) are as follows (assuming  $\Delta_{\mathcal{W}} \leq 2$  throughout). First, we can take the number of distinguished persons as parameter (note that we know NP-hardness of the cases where  $|\mathcal{W}^*| = 1$  or  $|\mathcal{M}^*| = 0$ ). Second, we can consider the number of blocking pairs as the parameter. We show fixed-parameter tractability for both parameterizations.

**Theorem 16.** *There is a fixed-parameter algorithm for the special case of SMC where each woman finds at most two men acceptable (i.e.,  $\Delta_{\mathcal{W}} \leq 2$ ), with parameter the number  $|\mathcal{W}_0^*| + |\mathcal{M}_0^*|$  of distinguished men and women left unmatched by some stable matching.*

To begin, we state Proposition 2 containing simple observations that are all trivial implications of the fact that each woman finds at most two men acceptable.

**Proposition 2.** *Let  $P_1$  and  $P_2$  be two augmenting paths.*

- (a) *If  $P_1$  and  $P_2$  start with the same edge, then one of them is a subpath of the other.*
- (b) *If  $P_1$  and  $P_2$  are not disjoint, then the set of their common vertices induces a suffix of either  $P_1$  or  $P_2$  (or both); their first common vertex is a man unless  $P_1$  and  $P_2$  start at the same woman.*
- (c) *If  $P_1$  and  $P_2$  are disjoint and  $e$  is an edge incident to both, then one of the paths starts or ends at a woman  $w$ , and  $e$  connects  $w$  with a man on the other path.*

Before describing the algorithm that proves Theorem 16, we introduce additional notation and make a couple of simple observations. To this end, let  $M^{\text{opt}}$  denote an optimal solution for our instance  $I$  such that  $M_s \triangle M^{\text{opt}}$  contains the minimum number of edges; recall that  $M_s$  is a fixed stable matching for  $I$ . Let  $b$  be the number of blocking pairs in  $M^{\text{opt}}$ .

We say that an edge  $f = \{m, w\}$  (with  $m \in \mathcal{M}$  and  $w \in \mathcal{W}$ ) is *dependent* if it connects two different connected components  $K_1$  and  $K_2$  of  $M_s \triangle M^{\text{opt}}$  and, in addition, it holds that  $M_s \triangle K_1$  admits more blocking pairs than  $M_s \triangle (K_1 \cup K_2)$ .

We will say that  $f$ , and with a slight abuse of the notation, also  $K_1$  *relies* on  $K_2$ . By claim (c) of Proposition 2, this is only possible in the following two scenarios, depicted in Fig. 6:

- $f$  has *type A*:  $w$  is the endpoint of  $K_2$  (which must be a path), is unmatched by  $M_s$  and prefers  $M^{\text{opt}}(w)$  to  $m$ , and  $f$  connects  $w$  with a man  $m$  on  $K_1$  that prefers  $M_s(m)$  to  $w$ , and  $w$  to  $M^{\text{opt}}(m)$ ;
- $f$  has *type B*:  $w$  is the endpoint of  $K_1$  (which must be a path), unmatched by  $M^{\text{opt}}$ , and  $f$  connects  $w$  with a man  $m$  on  $K_2$  that prefers  $M^{\text{opt}}(m)$  to  $w$ , and  $w$  to  $M_s(m)$ .

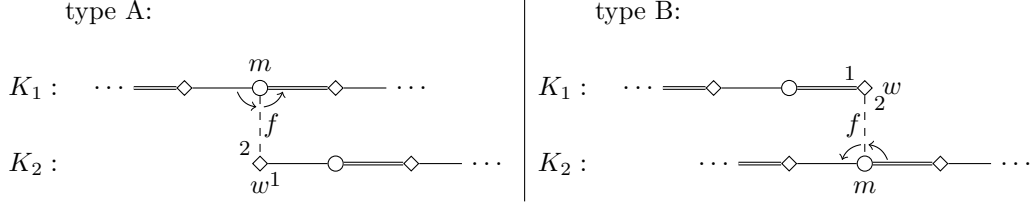


Figure 6: Illustration of a dependent edge  $f$ , running between two connected components  $K_1$  and  $K_2$  of  $M_s \triangle M^{\text{opt}}$  where  $K_1$  relies on  $K_2$ . Throughout the paper, we use squares for women, circles for men; distinguished persons are denoted by filled squares/circles. The numbering of edges incident to some vertex (or, sometimes, arrows between edges) indicate preferences. Further, double lines here denote edges of  $M_s$ , single lines denote edges of  $M^{\text{opt}}$ , and  $f$  is drawn with a dashed line.

**Lemma 17.** *All paths in  $M_s \triangle M^{\text{opt}}$  are augmenting paths. Further, if  $K_1$  and  $K_2$  are connected components of  $M_s \triangle M^{\text{opt}}$  such that  $K_1$  relies on  $K_2$  via a dependent edge  $f$ , then*

- (a) *if  $f$  has type A, then  $K_2$  is a masculine and not feminine path;*
- (b) *if  $f$  has type B, then  $K_1$  is a feminine path and  $K_2$  is either a cycle or a feminine path.*

*Proof.* For contradiction, let us first suppose that  $Q$  is a non-augmenting path in  $M_s \triangle M^{\text{opt}}$ , i.e., neither of its endpoints is distinguished. This implies that  $M_Q := M^{\text{opt}} \triangle Q$  is feasible. Recall that  $b$  is the number of blocking pairs  $M^{\text{opt}}$  admits. If  $M_Q$  admits at most  $b$  blocking pairs as well, then this contradicts to the choice of  $M^{\text{opt}}$ , because there are strictly less edges in  $M_s \triangle M_Q$  than in  $M_s \triangle M^{\text{opt}}$ .

Hence,  $M^{\text{opt}} \triangle Q$  admits at least  $b + 1$  blocking pairs. It is easy to see that since  $M_s$  is stable and  $Q$  is a path, there must be an edge along  $Q$  that blocks  $M^{\text{opt}}$ . By contrast, there is no edge on  $Q$  that blocks  $M_s$ , since  $M_s$  is stable. Hence, we know that modifying  $M^{\text{opt}}$  by switching the edges of  $M_s$  and  $M^{\text{opt}}$  along  $Q$  can only increase the number of blocking pairs if there are at least two dependent edges relying on  $Q$ . Clearly, one of these must have type B.

Let us call the man endpoint of a type B dependent edge a *joiner*; by the previous paragraph,  $Q$  contains at least one joiner. Let us fix an “outer-most” joiner  $m$  on  $Q$ . More precisely, we choose  $m$  so that the following holds: if  $m$  splits  $Q$  into two subpaths  $Q_1$  and  $Q_2$  with  $Q_1$  containing  $M^{\text{opt}}(m)$ , then  $Q_1$  contains no other joiners. Now, there might be several women who form a dependent edge with  $m$ , so let  $w$  denote the one that is most preferred by  $m$ . Let  $f$  be the edge  $\{m, w\}$ , and let  $P$  be the path of  $M_s \triangle M^{\text{opt}}$  that has  $w$  as its endpoint. We illustrate these concepts in Fig. 7.

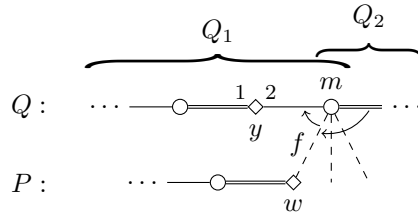


Figure 7: The joiner  $m$  splits  $Q$  into subpaths  $Q_1$  and  $Q_2$ . Double lines denote edges of  $M_s$ , single lines denote edges of  $M^{\text{opt}}$ , and dashed lines are for dependent edges.

We claim that  $M_f = M^{\text{opt}} \triangle (Q_1 \cup \{f\})$  is an optimal solution. Observe that  $M_s \triangle M_f$  can be obtained from  $M_s \triangle M^{\text{opt}}$  by deleting  $Q$  and substituting  $P$  by the path  $P + f + Q_2$  (where the plus sign means concatenation). Let  $x$  denote the endpoint of  $Q_1$  that is not  $m$ . First,  $M_f$  is clearly feasible, since  $x$  is not in  $W_0^*$ . Next, observe that by the stability of  $M_s$ , the only edge that may become blocking in  $M_f$  (and is not blocking in  $M^{\text{opt}}$ ) is a possible dependent edge of type A incident to  $x$ . If indeed there exists such an

edge, then  $x$  must be a woman not covered by  $M_s$ . Moreover, since  $f$  is a dependent edge of type B, we know that  $m$  prefers  $y = M^{\text{opt}}(m)$  to  $M_s(m)$ , and hence,  $y$  must prefer  $M_s(y)$  to  $m$ . However, it is then straightforward to check that  $Q_1$  must contain at least one edge that blocks  $M^{\text{opt}}$ , and this edge clearly is not blocking in  $M_f$ . (Note also the implication that  $Q_1$  has at least two edges.) Thus, the number of edges blocking  $M_f$  cannot be more than  $b$ . Hence,  $M_f$  is an optimal solution with less edges in  $M_s \triangle M_f$  than in  $M_s \triangle M^{\text{opt}}$ , a contradiction. This proves the first statement of the lemma.

Let us prove (b) now. Suppose that  $K_1$  and  $K_2$  are two connected components of  $M_s \triangle M^{\text{opt}}$  such that  $K_1$  relies on  $K_2$  via an edge  $f = \{w, m\}$  of type B. It is immediate that  $K_1$  cannot be a masculine path, so by the first statement of the lemma, it must be feminine. It remains to show that if  $K_2$  is path, then it is feminine. Assume for contradiction that  $K_2$  is a non-feminine path  $Q$  in  $M_s \triangle M^{\text{opt}}$  and some other path  $P$  in  $M_s \triangle M^{\text{opt}}$  relies on  $Q$  via a type B edge  $f$ . In this case we can argue exactly as above to show that there must exist an optimal matching  $M_f$  (defined the same way as we did while proving the first statement of the lemma) for which  $M_s \triangle M_f$  contains less edges than  $M_s \triangle M^{\text{opt}}$ , a contradiction.

To show (a), suppose that  $K_1$  and  $K_2$  are two connected components of  $M_s \triangle M^{\text{opt}}$  such that  $K_1$  relies on  $K_2$  via an edge  $f = \{w, m\}$  of type A. By the definition of a type A edge, the woman endpoint  $w$  of  $f$  is unmatched in  $M_s$ , and  $K_2$  is a path that has  $w$  as an endpoint. Also,  $m$  is the second choice of  $w$  and  $M^{\text{opt}}(w)$  is the first choice of  $w$ . Hence  $w \in \mathcal{W}_0^*$  is not possible, as in that case the edge  $f$  would have been deleted in Step 1 of the algorithm. Furthermore, since  $K_2$  is an  $M_s$ -alternating path with an endpoint in  $\mathcal{W}_0$ , the other endpoint of  $K_2$  cannot be a woman in  $\mathcal{W}_0$ . From this follows that  $Q_2$  is not a feminine augmenting path. Since (by the first statement of the lemma)  $K_2$  is an augmenting path, (a) follows.  $\square$

By the above lemma, each path in  $M_s \triangle M^{\text{opt}}$  is an augmenting path; we let  $P_x^{\text{opt}}$  denote the augmenting path in  $M_s \triangle M^{\text{opt}}$  that contains some  $x \in \mathcal{W}_0^* \cup \mathcal{M}_0^*$  as an endpoint.

We are now ready to present our algorithm, which is a branching algorithm: throughout its course, we make several “guesses” for which all possibilities have to be explored. When certain guesses turn out to be trivially wrong, such guesses are discarded, and we might not explicitly mention this in the algorithm. (In Step 1, we describe such issues in detail for illustration, but later we omit them.)

### Step 1: Guessing the first edges of augmenting paths.

First, for each  $w \in \mathcal{W}_0^*$  with  $|L(w)| = 2$ , we guess the edge of  $M^{\text{opt}}$  incident to  $w$ . This results in at most  $2^{|\mathcal{W}_0^*|}$  possibilities, all of which must be explored. Naturally, we discard those guesses where the edges  $\{w, M^{\text{opt}}(w)\}$ ,  $w \in \mathcal{W}_0^*$ , do not form a matching. From now on we assume that we know  $M^{\text{opt}}(w)$  for each  $w \in \mathcal{W}_0^*$ .

Additionally, we delete those edges  $\{m, w\}$  for which  $w \in \mathcal{W}_0^*$  and  $w$  prefers  $M^{\text{opt}}(w)$  to  $m$ . Such edges are neither needed in  $M^{\text{opt}}$ , nor can they block any matching that contains all the edges  $\{w, M^{\text{opt}}(w)\}$ ,  $w \in \mathcal{W}_0^*$ , guessed in this step.

### Step 2: Finding cycles in $M_s \triangle M^{\text{opt}}$ .

We make one more guess for each  $w \in \mathcal{W}_0^*$  by guessing whether  $P_w^{\text{opt}}$  relies on some cycle of  $M_s \triangle M^{\text{opt}}$  or not. Observe that if  $P_w^{\text{opt}}$  relies on some cycle  $C$ , then by Proposition 2, both  $P_w^{\text{opt}}$  and  $C$  can be found in time  $O(|V(P_w^{\text{opt}})| + |V(C)|)$  by simply following the longest alternating path starting with the edge  $\{w, M^{\text{opt}}(w)\}$ : the last person on this path  $Q$  must be a woman  $x$  incident to an edge  $\{x, m\}$  for which  $m$  is on  $Q$ , and the subpath of  $Q$  between  $m$  and  $x$  together with the edge  $\{x, m\}$  form the cycle  $C$ . For an illustration, see Fig. 8.

By the choice of  $M^{\text{opt}}$ , any cycle in  $M_s \triangle M^{\text{opt}}$  must be incident to at least one dependent edge of type B, relying on the cycle (recall that dependent edges of type A rely on paths). Furthermore, by Lemma 17 we also know that all paths relying on a cycle must be feminine paths. Hence, all cycles in  $M_s \triangle M^{\text{opt}}$  and all paths relying on them are found in this step.

### Step 3: Finding neutral paths.

In this step, for each  $w \in \mathcal{M}_0^*$  we guess whether  $w$  lies on a neutral path in  $M_s \triangle M^{\text{opt}}$ . Clearly, if  $w$  lies on a neutral path, then  $P_w^{\text{opt}}$  must be the unique longest augmenting path starting with the edge  $\{w, M^{\text{opt}}(w)\}$  by Proposition 2.

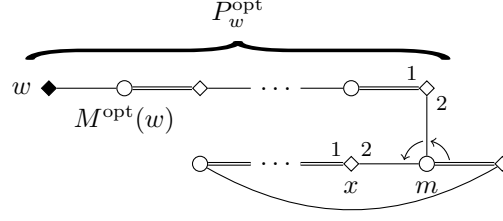


Figure 8: Step 2 of the algorithm for Theorem 16, for finding cycles in  $M \triangle M^{\text{opt}}$ . Double edges denote the stable matching  $M_s$ .

#### Step 4: Finding feminine paths relying on other feminine paths.

In this step, we first guess for each  $w \in \mathcal{W}_0^*$  whether  $P_w^{\text{opt}}$  relies on another feminine path, and if so, we also guess on which one. Supposing that, according to our guesses,  $P_w^{\text{opt}}$  relies on  $P_y^{\text{opt}}$  for some  $w$  and  $y$  in  $\mathcal{W}_0^*$ , we can find  $P_w^{\text{opt}}$  easily: by Lemma 17, we know that  $P_w^{\text{opt}}$  and  $P_y^{\text{opt}}$  are connected by a dependent edge  $f$  of type B, and by Proposition 2, there exists only one edge connecting two augmenting paths starting at different vertices, implying that  $f$  and  $P_w^{\text{opt}}$  can be found in  $O(|P_w^{\text{opt}}| + |P_y^{\text{opt}}|)$  time. To ensure that  $P_y^{\text{opt}}$  indeed contains the man  $m$  incident to  $f$ , we store  $m$  as an *obligatory* man for  $y$ .

#### Step 5: Finding all remaining feminine paths.

Let  $\mathcal{W}_r$  be the set of those distinguished women  $w$  for which  $P_w^{\text{opt}}$  has not been found yet. For each  $w \in \mathcal{W}_r$ , we define two paths. First, we let  $Q_w^1$  be the shortest augmenting path starting with the edge  $\{w, M^{\text{opt}}(w)\}$  that contains all obligatory men for  $w$  and is *valid*, meaning that it does not end at a distinguished person. Second, let  $Q_w^2$  be the shortest valid augmenting path starting with the edge  $\{w, M^{\text{opt}}(w)\}$  and ending at a vertex  $x$  such that  $M_s \triangle Q_w^2$  admits less blocking pairs than  $M_s \triangle Q_w^1$ .

**Lemma 18.** *If  $P_w^{\text{opt}}$  does not rely on any other path or cycle in  $M_s \triangle M^{\text{opt}}$ , then either  $P_w^{\text{opt}} = Q_w^1$  or  $P_w^{\text{opt}} = Q_w^2$ .*

*Proof.* First observe that for any  $w \in \mathcal{W}_r$ , if the guesses made by the algorithm are correct, then  $P_w^{\text{opt}}$  contains  $Q_w^1$  as a subpath. To see this, recall that if the guesses made in Step 4 of the algorithm are correct, then  $P_w^{\text{opt}}$  indeed must contain all obligatory men for  $w$ . Also,  $P_w^{\text{opt}}$  must be valid by the feasibility of  $M^{\text{opt}}$ , proving that  $P_w^{\text{opt}}$  indeed contains  $Q_w^1$ .

Next, suppose that  $P_w^{\text{opt}} \neq Q_w^1$ , and consider the matching  $M_1 = (M^{\text{opt}} \triangle P_w^{\text{opt}}) \triangle Q_w^1$  (in other words, replace  $P_w^{\text{opt}}$  with  $Q_w^1$  in the difference of  $M_s$  and the solution). By Lemma 17, only feminine paths may rely on  $P_w^{\text{opt}}$ . Since  $Q_w^1$  contains all the obligatory men for  $w$ , we get that all paths that rely on  $P_w^{\text{opt}}$  in  $M_s \triangle M^{\text{opt}}$  rely on  $Q_w^1$  in  $M_s \triangle M_1$  as well (assuming correct guesses in Step 4). Using this and the stability of  $M_s$ , it follows that an edge that is blocking in  $M_1$  but is not blocking in  $M^{\text{opt}}$  must be incident to the last person  $x$  on  $Q_w^1$ ; note that  $x$  is a woman.

Let  $c_1$  denote the number of blocking pairs with respect to  $M_1$  that are incident to a man or woman on  $Q_w^1$ , and let  $c^{\text{opt}}$  be the number of pairs with respect to  $M^{\text{opt}}$  that are incident to a man or woman on  $P_w^{\text{opt}}$ . We claim  $c^{\text{opt}} = c_1 - 1$ . Clearly,  $c^{\text{opt}} < c_1$ , as otherwise  $M_1$  would also be an optimal solution which would contradict our choice of  $M^{\text{opt}}$  (as  $M_s \triangle M_1$  contains less edges than  $M_s \triangle M^{\text{opt}}$ ). From this, we immediately have that  $M_s \triangle P_w^{\text{opt}}$  admits less blocking pairs than  $M_s \triangle Q_w^1$ . On the other hand,  $c^{\text{opt}} \leq c_1 - 2$  can only happen if both edges incident to  $x$  are blocking in  $M_1$ , and none of them is blocking in  $M^{\text{opt}}$ . However, this cannot happen, as we are going to show now. Let  $m_1 = M_s(x)$  and  $m_2 = M^{\text{opt}}(x)$  be the two men in  $x$ 's preference list (note that  $Q_w^1$  contains  $m_1$  but not  $m_2$ ). If both  $\{x, m_1\}$  and  $\{x, m_2\}$  are blocking in  $M_1$ , then  $m_1$  prefers  $x$  to  $M^{\text{opt}}(m_1)$ , and  $m_2$  prefers  $x$  to  $M_s(m_2)$ . The latter (and the stability of  $M_s$ ) implies that  $x$  prefers  $m_1$  to  $m_2$ . But this shows that  $\{x, m_1\}$  is a blocking pair in  $M^{\text{opt}}$  as well, proving our claim.

Therefore, the number of blocking pairs incident to  $P_w^{\text{opt}}$  in  $M^{\text{opt}}$  is only one less than the number of blocking pairs incident to  $Q_w^1$  in  $M_1$ . Also, by the arguments of the previous paragraph, choosing an augmenting path that is even longer than  $Q_w^2$  cannot further decrease the number of blocking pairs. Hence, by the choice of  $M^{\text{opt}}$ , we get that  $P_w^{\text{opt}}$  indeed must be equal to  $Q_w^2$ . This proves the lemma.  $\square$

According to Lemma 18, we can find all remaining feminine paths in  $M_s \triangle M^{\text{opt}}$ , by guessing for each  $w \in \mathcal{W}_r$  whether  $P_w^{\text{opt}}$  equals  $Q_w^1$  or  $Q_w^2$ .

#### Step 6: Computing elimination paths.

Let  $\mathcal{F}$  be the union of cycles, feminine and neutral paths found in Steps 1 to 5. When searching for masculine paths, we will have to deal with edges that might be type A dependent edges in the optimum solution. We call an edge *volatile* if it connects a woman in  $\mathcal{W}_0$  with its second choice. The importance of this definition is showed by the following lemma.

**Lemma 19.** *If  $f$  is a volatile edge incident to some non-feminine path in  $M_s \triangle M^{\text{opt}}$ , then  $f$  is not blocking in  $M^{\text{opt}}$ .*

*Proof.* Recall that  $M^{\text{opt}}$  is an optimal solution for which  $M_s \triangle M^{\text{opt}}$  has as few edges as possible. Without loss of generality, we can further assume that there does not exist another optimal solution  $M'$  with the same number of edges in  $M_s \triangle M^{\text{opt}}$  in which for each man  $m$ , either  $M'(m) = M^{\text{opt}}(m)$  or  $m$  prefers  $M'(m)$  to  $M^{\text{opt}}(m)$  (as otherwise we can pick  $M'$  instead of  $M^{\text{opt}}$ ).

Suppose, for sake of contradiction, that  $f$  is a volatile edge incident to a masculine path  $P_x^{\text{opt}}$  in  $M_s \triangle M^{\text{opt}}$  and  $f$  blocks  $M^{\text{opt}}$ . Let  $m$  and  $w$  be the man and woman connected by  $f$ . First observe that  $w$  does not lie on  $P_x^{\text{opt}}$ , as in that case  $w$  would get its first choice by  $M^{\text{opt}}$ , and  $f$  would not be blocking. Therefore,  $m$  lies on  $P_x^{\text{opt}}$ . Since  $f$  is blocking in  $M^{\text{opt}}$ , we know that  $M^{\text{opt}}$  does not cover  $w$ .

Define  $P$  as the subpath of  $P_x^{\text{opt}}$  from  $x$  to  $m$  plus the edge  $f$ . Consider now the matching  $M_f = M^{\text{opt}} \triangle (P_x^{\text{opt}} \triangle P)$ . Clearly,  $M_f$  is feasible, since  $P_x^{\text{opt}}$  is not a feminine path. Furthermore,  $f$  is not blocking in  $M_f$ , and there can be at most one new blocking edge  $f'$  in  $M_f$  that does not block  $M^{\text{opt}}$ . Moreover, if such an edge  $f'$  exists, then  $P_x^{\text{opt}}$  must end at some woman  $w'$  in  $\mathcal{W}_0$  and  $f'$  must be a volatile edge incident to  $w'$ . Let us call this modification the *filtering* of  $f$ ; observe that the modified matching admits at most as many blocking pairs as  $M^{\text{opt}}$ .

Let us repeat the above procedure until it is no longer applicable: while there exists a volatile edge  $e$  that is blocking in the actual matching  $M$  but was not blocking in  $M^{\text{opt}}$  (there can be only one such  $e$ ), filter  $e$  if it is incident to a non-feminine masculine path in  $M_s \triangle M$ , and stop otherwise. Clearly, this process terminates (as at each step we either make a masculine paths shorter, or match its last man with a better partner). In the end, we clearly arrive at an optimal matching that has either more common edges with  $M_s$  than  $M^{\text{opt}}$ , or in which all men are as happy as in  $M^{\text{opt}}$ , both cases contradicting our choice of  $M^{\text{opt}}$ .  $\square$

For any volatile edge  $f$ , we can decide in linear time whether there exists a valid masculine augmenting path disjoint from  $\mathcal{F}$  that contains the woman endpoint of  $f$  but does not contain  $f$  itself. If such a path exists, then it is unique by Proposition 1 and we denote it by  $Q_f$ . Let  $f$  be a volatile edge that is blocking in  $M_s \triangle \mathcal{F}$ . We say that a set  $\mathcal{P}_f$  of masculine augmenting paths *eliminates*  $f$ , if: (i)  $Q_f$  exists and  $Q_f \in \mathcal{P}_f$ , and (ii) for any path  $Q \in \mathcal{P}_f$ , if there is a volatile blocking edge  $f'$  in  $M_s \triangle Q$ , then  $Q_{f'}$  exists and is contained in  $\mathcal{P}_f$ . We refer to the (inclusion-wise) minimal set of masculine paths that eliminates  $f$  as the *elimination paths* for  $f$ , and we denote it by  $\mathcal{P}_f^{\text{elim}}$ . Further, we refer to the starting points of these paths as the *elimination set* for  $f$ .

#### Step 7: Guessing relevant elimination sets in $M^{\text{opt}}$ .

We call an edge *relevant* in  $M^{\text{opt}}$ , if it is a volatile edge blocking  $M_s \triangle \mathcal{F}$ , but it does not block  $M^{\text{opt}}$ . By Lemma 19 and the definition of elimination sets, we know that if  $f$  is a relevant edge in  $M^{\text{opt}}$ , then  $M_s \triangle M^{\text{opt}}$  must contain all paths in  $\mathcal{P}_f^{\text{elim}}$ , to ensure that there are no volatile edges incident to masculine paths that block  $M^{\text{opt}}$ . Since there may be several volatile edges blocking in  $M_s \triangle \mathcal{F}$ , we cannot determine the relevant ones among them by simply guessing them. Instead, we only guess the elimination sets for all relevant edges. Clearly, these sets must be pairwise disjoint subsets of  $\mathcal{M}_0^*$ , let  $R_1, \dots, R_\ell$  denote them.

#### Step 8: Computing cheapest masculine paths.

First, for each set  $R_i \subseteq \mathcal{M}_0^*$  with  $i = 1, \dots, \ell$  that, according to our guesses made in Step 7, forms the elimination set for some volatile edge  $f$  relevant in  $M^{\text{opt}}$ , we determine *some* volatile edge  $f$  incident to  $\mathcal{F}$  that is blocking in  $M_s \triangle \mathcal{F}$  and whose elimination set is exactly  $R_i$ . Namely, we pick an edge  $f$  among all



such edges in a way that the number of blocking pairs in  $M_s \Delta (\mathcal{F} \cup \mathcal{P}_f^{\text{elim}})$  is as small as possible. Let  $f_i$  be the volatile edge chosen this way, and let  $\mathcal{P}^{\text{elim}} = \bigcup_{1 \leq i \leq \ell} \mathcal{P}_{f_i}^{\text{elim}}$ .

Next, let  $\mathcal{M}_r = \mathcal{M}_0^* \setminus (R_1 \cup \dots \cup R_\ell)$  be the set of those distinguished men that are not covered by  $M_s$  and are not contained in any of the elimination sets  $R_1, \dots, R_\ell$ . It remains to determine a valid augmenting path for each man  $m \in \mathcal{M}_r$ . Therefore, for each such  $m$  we compute a valid augmenting path  $P_m$  disjoint from  $\mathcal{F}$  such that the number of blocking pairs in  $M_s \Delta \mathcal{P}_r$  is minimized, where  $\mathcal{P}_r$  is the union of all paths  $P_m$  for some  $m \in \mathcal{M}_r$ . We perform this by using the algorithm of Theorem 11, switching the roles of men and women, and restricting the algorithm to consider only valid masculine augmenting paths disjoint from  $\mathcal{F}$ .

Finally, we output the matching  $M^{\text{out}} = M_s \Delta (\mathcal{F} \cup \mathcal{P}^{\text{elim}} \cup \mathcal{P}_r)$ .

It is straightforward to verify that the number of guesses made are bounded by a function of  $|\mathcal{W}_0^*| + |\mathcal{M}_0^*|$ , and all computations in a branch can be performed in time polynomial in the size  $|I|$  of the instance, yielding a fixed-parameter algorithm.

We now prove the correctness of the above algorithm.

*Proof of Theorem 16.* To prove the correctness of the proposed algorithm, we first show that if all our guesses are true, then the paths and cycles in  $\mathcal{F}$  are exactly the feminine paths and the cycles of  $M_s \Delta M^{\text{opt}}$ . From the description of our algorithm, it should be clear that the correctness of Steps 2, 3, and 4 follows directly from Proposition 2 and Lemma 17. Lemma 18 guarantees the correctness of Step 5, which proves that in Steps 1-5 the algorithm indeed finds all cycles and feminine paths of  $M_s \Delta M^{\text{opt}}$ .

Next, let us argue that  $M^{\text{out}}$  is indeed a matching. For this, apart from the correctness of Steps 1 to 5, we need that the masculine paths in  $M_s \Delta M^{\text{out}}$  are disjoint from  $\mathcal{F}$ . In addition, we also need that paths in  $\mathcal{P}^{\text{elim}}$  are disjoint from all remaining masculine paths. To see this, observe that each such path  $P$  ends at a woman  $w \in \mathcal{W}_0$  which is connected by a volatile edge (not on  $P$ ) to either  $\mathcal{F}$  or to another path in  $\mathcal{P}^{\text{elim}}$ . Hence,  $w$  cannot lie on any masculine path other than  $P$  by Proposition 1. This shows that  $M^{\text{out}}$  is a matching. Its feasibility is implied by the correctness of Steps 1 to 5, and the facts that the algorithm only chooses valid masculine augmenting paths.

We argue that  $M^{\text{out}}$  admits at most as many blocking pairs as  $M^{\text{opt}}$ . First, observe that all edges blocking in  $M_s \Delta \mathcal{F}$  are either relevant volatile edges in  $M^{\text{opt}}$ , or they are also blocking in  $M^{\text{opt}}$ . Assuming our guesses in Step 7 are correct, there are exactly  $\ell$  relevant volatile edges in  $M^{\text{opt}}$ . Furthermore, if  $e_i$  is a relevant edge with elimination set  $R_i$  for some  $i \in \{1, \dots, \ell\}$ , then by Lemma 19 we know that all elimination paths in  $\mathcal{P}_{e_i}^{\text{elim}}$  must be contained in  $M_s \Delta M^{\text{opt}}$ . Hence, in Step 8 the algorithm is bound to find *some* volatile edge  $f_i$  (though not necessarily  $e_i$ ), that is blocking in  $M_s \Delta \mathcal{F}$  and whose elimination set is  $R_i$ . Clearly, by the definition of elimination paths,  $f_i$  is not blocking in  $M^{\text{out}}$ . Thus, there are at least  $\ell$  volatile edges blocking in  $M_s \Delta \mathcal{F}$  but not blocking in  $M^{\text{out}}$ .

It remains to count the number of blocking pairs in  $M^{\text{out}}$  adjacent only to masculine paths. First, by our choice of  $e_i$ , there are at most as many blocking pairs in  $M^{\text{out}}$  incident to paths in  $\mathcal{P}_{f_i}^{\text{elim}}$ , as there are in  $M^{\text{opt}}$  incident to paths in  $\mathcal{P}_{e_i}^{\text{elim}}$ . Therefore, the number of blocking pairs in  $M_s \Delta (\mathcal{F} \cup \mathcal{P}^{\text{elim}})$  is at most the number of blocking pairs in  $M_s \Delta (\mathcal{F} \cup \bigcup_{1 \leq i \leq \ell} \mathcal{P}_{e_i}^{\text{elim}})$ .

Second, by the correctness of the algorithm of Theorem 11 the total number of blocking pairs in  $M_s \Delta \mathcal{P}_r$  is at most the number of blocking pairs in  $M^{\text{opt}}$  incident to the paths  $P_m^{\text{opt}}$ ,  $m \in \mathcal{M}_r$ . Observe that no augmenting path in  $M_s \Delta M^{\text{opt}}$  that does *not* start in  $\mathcal{M}_r$  can rely on a path  $P_m^{\text{opt}}$  for some  $m \in \mathcal{M}_r$ , so we can indeed choose augmenting paths starting from the men in  $\mathcal{M}_r$  independently from the rest of the solution. The optimality of  $M^{\text{out}}$  follows.  $\square$

The following corollary of Theorem 16 is implied by the observation that the number of blocking pairs admitted by  $M^{\text{opt}}$  is at least  $(|\mathcal{W}_0^*| + |\mathcal{M}_0^*|)/2$ , since each augmenting path contains at least one edge that blocks  $M^{\text{opt}}$ .

**Corollary 20.** *There is a fixed-parameter algorithm for the special case of SMC where each woman finds at most two men acceptable, with parameter  $b$ .*

## 7 Discussion

We provided a systematic study of the computational complexity of STABLE MARRIAGE WITH COVERING CONSTRAINTS. Our main result is a complete computational complexity trichotomy into polynomial-time solvable cases, fixed-parameter tractable cases, and W[1]-hard cases for five natural parameters: the number of distinguished men/women, the maximum length of preference lists for men/women, and the number of blocking pairs allowed. Thereby, we solved a problem by Hamada et al. [12].

Given the strong polynomial-time inapproximability bounds, as well as the parameterized intractability results of this paper, we pose as an open question whether *fixed-parameter approximation algorithms* can beat either of these obstacles for solving SMC. Another challenge for future research is to investigate possible adaptations of the proposed algorithms to the Hospitals/Residents model (naturally, all our hardness results for SMC-1 apply to the HRLQ problem).

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## A Decision Diagram for SMC

On the next page we provide a decision diagram showing that our results fully determine the computational complexity of SMC with respect to the set  $S = \{b, |\mathcal{W}^*|, |\mathcal{M}^*|, \Delta_{\mathcal{M}}, \Delta_{\mathcal{W}}\}$  of possible parameters. Going through this decision diagram should convince the reader that for any possible choice of the elements in  $S$  regarded as “constant”, “parameter” or “unbounded”, the presented results indeed classify the computational complexity of SMC as either polynomial-time solvable (P) or NP-hard, and in the latter case, either fixed-parameter tractable (FPT), or W[1]-hard with the given parameterization (if any). In particular, when we provide parameterized results, this means that the restriction of SMC in question is NP-hard without parameterization.

Before presenting our decision diagram, we remark that if a certain restriction of SMC where one of the values  $v \in S$  is a constant  $k$  proves to be NP-hard or W[1]-hard with some parameterization, then it is easy to see that the same hardness result also holds for the case where  $v \geq k$  (and all other assumptions are the same).

Finally, we will sometimes use the “reflection” of a result, by which we mean switching the roles of men and women. We refer to the reflection of a result by adding the postfix ‘R’ to its name, so Theorem  $xR$  denotes the reflection of Theorem  $x$ .

